Identifiability in matrix sparse factorization

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Abstract

Matrix sparse factorization is a multilinear inverse problem where given an observed matrix $\boldsymbol{Z}$ and some sparsity constraints, one tries to recover some sparse factors for which the matrix product is equal to $\boldsymbol{Z}$. In order to better understand how to design provably good algorithms for matrix sparse factorization, this work provides some *identifiability* results in the case of matrix sparse factorization with only two factors, *i.e.*, some conditions for which the observation $\boldsymbol{Z}$ is sufficient to recover without ambiguity the pair of sparse factors $(\boldsymbol{X}, \boldsymbol{Y})$ for which $\boldsymbol{XY} = \boldsymbol{Z}$, up to unavoidable permutation and scaling ambiguities due to the nature of matrix product. In particular, this work analyzes two important problem variations: the case where one of the two factors is fixed, and the case where the support of each factor is fixed. In the first case, identifiability of the right factor when the left factor is fixed can be characterized by using linear independence of specific subsets of columns in the fixed left factor. In the second case, identifiability with a fixed pair of supports can be characterized by *iterative completability*, *i.e.*, the fact that the rank 1 matrices induced by the product between one column of the left factor and one row of the right factor can be completed one by one. Characterization of identifiability in these two specific problem variations allows us to establish some important necessary conditions for identifiability in the general case, which can lead in a future work to general conditions of identifiability in matrix sparse factorization.

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Contents

Chapter 1

Introduction

The topic of this internship is to explore identifiability issues in matrix sparse factorization, which can be seen as a multilinear inverse problem. Essentially, given an observed matrix $\boldsymbol{Z}$ and some sparsity constraints, we want to find some conditions which guarantees that the sparse factorization of $\boldsymbol{Z}$ is unique, up to some natural equivalence relation.

1.1 Context and motivation

In several research domains like computer vision, natural language processing, etc., many state-of-the-art results have been achieved by learning with deep neural networks. However, such performance comes at the cost of requiring a huge amount of computing resources during the learning phase. One way to reduce the complexity of deep neural networks is to enforce some sparsity constraints on the number of connections between layers: only a few parameters of the model are nonzero. Such sparse models, so-called *sparse neural networks* [13, 23], allows for faster computation of the model's output given an input, and also for lesser memory storage of the model. Although [13] shows the possibility to retrain a sparse neural network in isolation starting from a trained non-sparse neural network, it is not clear how to train sparse neural networks directly from scratch. In order to better understand how to design algorithms which promote sparsity in deep neural networks, one can start by considering linear neural networks, and then try to generalize the obtained results for nonlinear neural networks.

Since the realization of a linear neural network with L layers is a linear operator which can be factorized into L matrices, removing some connections between layers in the linear neural network is equivalent to enforcing sparsity constraints on the factors in the matrix factorization. One approach to deal with the problem of matrix sparse factorization is to see it as a *multilinear inverse problem*. Indeed, the matrix product is the multilinear operator $(\boldsymbol{X_l})_{l=1}^L \mapsto \prod_{l=1}^L \boldsymbol{X_l}$. Given an observed matrix $\boldsymbol{Z}$, and some sparsity constraint sets $(\mathcal{E}_l)_{l=1}^L$, factorizing $\boldsymbol{Z}$ into L sparse factors is the following multilinear inverse problem:

$$\begin{aligned} \text{find} \quad & (\boldsymbol{X_l})_{l=1}^L \\ \text{such that} \quad & \prod_{l=1}^L \boldsymbol{X_l} = \boldsymbol{Z}, \\ & \boldsymbol{X_l} \in \mathcal{E}_l \text{ is true } \forall l \in \{1, ..., L\}. \end{aligned} \tag{1.1}$$

For instance, given some integers $(s_l)_{l=1}^L$, we can choose $\mathcal{E}_l := \{\boldsymbol{X} \mid \|\boldsymbol{X}\|_0 \leq s_l\}$ as the sparsity constraint sets for each $l \in \{1, ..., L\}$, where $\|\cdot\|_0$ counts the number of nonzero entries in the matrix.

A key issue in this inverse problem is the one of *identifiability*: "given an observation, supposing there exists a solution to the inverse problem, is this solution unique?" Identifiability for *linear* inverse problems with sparsity constraints has been largely studied in the compressive sensing literature [12]. In compressive sensing, one considers a linear inverse problem $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$, in which the unknown representation $\boldsymbol{x} \in \mathbb{C}^N$ is recovered from the measurement $\boldsymbol{y} \in \mathbb{C}^m$, and where $\boldsymbol{A} \in \mathbb{C}^{m \times N}$ is the matrix modeling the linear measurement process. In the case where the number of measures m is smaller than the dimension of the representation N, the linear system is underdetermined, and leads to infinitely many solutions, provided that there exists a solution. However, when enforcing some sparsity constraints on the representation $\boldsymbol{x}$, it is possible to reconstruct the representation $\boldsymbol{x}$ from the signal $\boldsymbol{y}$, with efficient algorithms. In fact, it is quite well understood under which conditions the signal can be recovered without ambiguities, which makes the design of reconstruction algorithms possible with performance guarantees. The restricted isometry property [7, 12] is an example of such conditions.

In contrast, characterization of identifiability for *multilinear* inverse problems with sparsity constraints is still lacking in the literature. By analogy with the linear case, understanding under which conditions the multilinear inverse problem (1.1) admits a unique solution up to equivalence relations could be critical to understand how to design provably good algorithms for solving the following optimization problem introduced in [20]:

$$\min_{\boldsymbol{X_1},\ldots,\boldsymbol{X_L}} \|\boldsymbol{Z} - \prod_{l=1}^{L} \boldsymbol{X_l}\|^2 + \sum_{l=1}^{L} \delta_{\mathcal{E}_l}(\boldsymbol{X_l}) \tag{1.2}$$

where $(\delta_{\mathcal{E}_l})_{l=1}^L$ are sparsity inducing penalties enforcing the sparsity constraints on the factors $(\boldsymbol{X_l})_{l=1}^L$, given the sparsity constraint sets $(\mathcal{E}_l)_{l=1}^L$. This problem is important, because solving it efficiently and robustly could allow us to find a fast transform for any linear operator. In [20], the authors proposed a proximal algorithm to solve (1.2), which can factorize for instance the Hadamard transform matrix into sparse factors. However, for a general matrix $\boldsymbol{Z}$, it is not clear under which conditions the optimization is successful. Depending on the initialization of the algorithm, the optimization might fail in the matrix sparse factorization.

1.2 Objective

Therefore, the main motivation of this work is to characterize identifiability for matrix sparse factorization, in order to understand how to design provably good algorithms for matrix sparse factorization. This work will present some identifiability results in matrix sparse factorization, in the specific case where only two layers are considered. This simplification is justified by a hierarchical approach used in [20] to approximate a given matrix $\boldsymbol{Z}$ by a product of sparse factors, in which the proposed algorithm iteratively factorizes the input matrix $\boldsymbol{Z}$ into two factors, one being sparse and the other less sparse. The process is then repeated on the less sparse factor, until the desired number of factors L is obtained. Ideally, we hope that after understanding the case with two layers, we can generalize the obtained results in the case with several layers.

Then, when $L = 2$, the multilinear inverse problem in (1.1) becomes a *bilinear* inverse problem, where the bilinear mapping is simply the matrix product of two factors $(\boldsymbol{X}, \boldsymbol{Y}) \mapsto \boldsymbol{XY}$. The objective is to understand under which conditions (1.1) has a unique solution for the case $L = 2$. However, because of the nature of the matrix product operator, it is necessary to take into account natural equivalence relations between pairs of factors, like scaling or permutation equivalence. The correct problem formulation, presented in Section 2.2, will take into account such equivalence relations.

1.3 Related works

The literature already provides some identifiability results in general bilinear inverse problems, but specific results for matrix sparse facorization is lacking. In particular, we are looking for general conditions of identifiability in matrix sparse factorization which are also easy to verify in practice.

1.3.1 Model-based compressive sensing

As presented in Section 1.1, compressive sensing aims to recover a sparse representation of a signal, given a measurement matrix. In contrast to traditional compressive sensing algorithms which only consider simple sparsity (defined as the number of nonzero entries), model-based compressive sensing introduced in [3] considers structured sparsity models that encode the inter-dependency between nonzero entries in the representation vector, which reduces its degrees of freedom. More precisely, the additional structure enforced by the sparsity constraints leads to the definition of the set of sparse vectors as a union of specific subspaces [19, 4]. Then, by leveraging such sparsity structures, model-based compressive sensing algorithms can perform better compression during the recovery. For instance, in [3], it has been shown experimentally that using wavelet tree models [10] or block-sparse models [8], model-based recovery requires less measurements than standard recovery. In matrix sparse factorization, because of the nature of matrix product, it might be natural to favor some adapted sparsity structure for the factors. Understanding the nature of these structured sparsity models in matrix sparse factorization will help us in better understanding the guarantees for performing robust and efficient matrix sparse factorization. In other words, considering well-chosen structured sparsity models might be the key to designing efficient algorithms from matrix sparse factorization, because such algorithms can exploit the structure of these sparsity models. This is why in our work we will also use structured sparsity models, and introduce for this purpose the notion of *family of allowed support* in Section 2.1.2. Example of structures that has been considered in the literature for matrix sparse factorization is the permuted striped block model [22] further presented in the next paragraph.

1.3.2 Structured sparsity model in matrix sparse factorization

Some natural structured sparsity model for studying identifiability in matrix sparse factorization has been introduced in the literature. In [15, Chapter 7], the sparsity model considered in the case of matrix sparse factorization with two factors is the model where the left factor has k-sparse columns (at most k nonzero entries per column), and the right factor has l-sparse rows (at most l nonzero entries per row). Then, given some assumptions on k, l and the size of matrix factors, the author showed that the butterfly factorization [18, 11] is the unique sparse factorization (up to equivalence) of the Discrete Fourier Transform (DFT) matrix. In other words, the butterfly factorization of the DFT matrix is identifiable for the considered structured sparsity model. However, the proof given in [15, Chapter 7] for the identifiability of the butterfly factorization depends on the entries of the DFT matrix, which means that identifiability results obtained from the author are specific to the instance of DFT matrix. In our work, we will try to establish identifiability results which does not depend on a specific instance of matrix factors. As it will be detailed in Chapter 2, this kind of identifiability results will be referred to as *global identifiability* results, in contrast to *instance identifiability* results.

Similarly to the sparsity structure discussed in the previous paragraph, [22] defines Permuted Striped Block (PSB) matrices, which are matrices composed of a sum of n rank 1 matrices of the form $\boldsymbol{a}_i\tilde{\boldsymbol{x}}_i^T$ where $\boldsymbol{a}_i \in \{0,1\}^m$ is a binary column vector with exactly $d \leq m$ nonzero entries, and $\tilde{\boldsymbol{x}}_i \in \mathbb{R}^N$ is a real vector with $k \leq N$ nonzero entries. Then, by adopting a probabilistic point of view where a probability distribution on PSB matrices is considered, [22] shows that, under some conditions, it is possible to factorize with high probability PSB matrices sampled from this distribution, and the obtained factorization is unique up to natural equivalences. However, in our work, we will not limit ourselves to binary matrices for the left factor, and we will focus on a deterministic point of view (instead of a probabilistic point of view).

1.3.3 Identifiability analysis using the lifting principle

In the literature, one major approach to analyze identifiability in bilinear inverse problems is to use the *lifting* principle, inspired from the optimization literature [2]. In [9], the authors proposed a general framework to transform any bilinear inverse problem into a matrix rank minimization problem subject to linear equality constraints, involving a linear operator called the lifting operator $\mathscr{L}$, defined on a matrix space, and determined by the original bilinear operator. Then, [9] establishes some identifiability results using the so-called *rank 2 null space* of this linear operator, which is the null space of this operator intersected with the set of matrices of rank at most 2. More precisely, the triviality of the rank 2 null space of this linear operator intersected with a so-called *secant set* [5] is a necessary and sufficient condition of identifiability for the bilinear inverse problem (in our case, the secant set is determined by the sparsity constraints). However, characterizing this restricted rank 2 null space is challenging for an arbitrary linear operator $\mathscr{L}$. In [9], the restricted rank 2 null space has been partly characterized for the specific case of blind deconvolution, but a general characterization is missing for matrix factorization. In this work, we will present in Chapter 4 an alternative linear operator $\mathscr{S}$ similar to $\mathscr{L}$ but more adapted to the specific instance of matrix factorization.

This lifting principle has the advantage to be easily extended to the case of multilayer sparse factorization with more than two factors [21]. Based on the fact that each entry of the matrix product $\boldsymbol{Z} = \prod_{i=1}^{L} \boldsymbol{X_l}$ is a sum of monomials of degree L involving terms from each factor $\boldsymbol{X_l}$, [21] used the lifting principle to formulate necessary and sufficient conditions of identifiability using a linear operator $\mathcal{A}$ determined by the supports of the factors. However, these conditions are also difficult to verify in practice, since most of the time it is difficult to give a closed-form expression of the linear operator $\mathcal{A}$, and only evaluations of $\mathcal{A}$ on rank 1 tensors can be computed. In our work, we will focus on the specific case of two factors in the matrix factorization, and try to characterize in a simple way the obtained conditions from the lifting procedure. Typically, we will propose a characterization of the triviality of the rank 2 null space of $\mathscr{S}$ intersected with the secant set by using some matrix completability conditions, as we will see in Section 1.3.6.

1.3.4 Identifiability analysis with individual identifiability

Consider in this paragraph a generic bilinear inverse problem, for a given bilinear operator $\boldsymbol{L}$: $(\boldsymbol{x}, \boldsymbol{y}) \mapsto \boldsymbol{L}(\boldsymbol{x}, \boldsymbol{y})$. An alternative approach to analyze identifiability in this kind of generic bilinear inverse problem is to divide identifiability of a solution $(\boldsymbol{x}, \boldsymbol{y})$ into identifiability of $\boldsymbol{x}$ and $\boldsymbol{y}$ individually. In [16], the author expanded the notion of identifiability by allowing uniqueness up to a group of transformations, and derives some general necessary and sufficient conditions for identifiability in bilinear inverse problems up to a transformation group, which essentially have the following interpretation: to identify a solution $(\boldsymbol{x}, \boldsymbol{y})$ up to a transformation group in a bilinear inverse problem, it is necessary and sufficient to prove that the left solution $\boldsymbol{x}$ is identifiable up to the transformation group, and that the right solution $\boldsymbol{y}$ is identifiable when the identified left solution is fixed. In our work, we consider the specific instance of matrix sparse factorization, and in addition to scaling ambiguities, we will take into account permutation ambiguities for pairs of factors in matrix product. The group of transformation considered in our work is the group of so-called *scaled permutations*:

$$\mathcal{C} := \{\boldsymbol{PD} \mid \boldsymbol{P} \text{ a permutation matrix}, \boldsymbol{D} \text{ a diagonal matrix with nonzero diagonal entries}\}, \tag{1.3}$$

and two pairs of factors $(\boldsymbol{X}, \boldsymbol{Y})$ and $(\boldsymbol{X'}, \boldsymbol{Y'})$ are equivalent if there exists $\boldsymbol{C} \in \mathcal{C}$ such that $\boldsymbol{XC} = \boldsymbol{X'}$ and $\boldsymbol{C}^{-1}\boldsymbol{Y} = \boldsymbol{Y'}$. Then, our analysis of identifiability in matrix sparse factorization will be inspired by the identifiability result in [16], in the sense that we will also divide the identifiability of a pair of factors $(\boldsymbol{X}, \boldsymbol{Y})$ into the identifiability of the left factor $\boldsymbol{X}$ and the right factor $\boldsymbol{Y}$ individually (see Section 2.3). In Chapter 3, we will present a simple characterization of *right identifiability*, which is the property of identifying the right factor when the left factor is fixed, using linear independence of specific subsets of columns in the fixed left factor.

1.3.5 Identifiability in other bilinear inverse problems

Based on the lifting principle presented in Section 1.3.3, several identifiability results have been proposed in blind deconvolution [17, 1] or phase retrieval [6]. However, identifiability results using the lifting principle in the specific case of matrix sparse factorization are still lacking in the literature. Therefore, this work aims to propose some identifiability results in matrix sparse factorization, based on existing frameworks for studying identifiability.

1.3.6 Identifiability results using matrix completability

Identifiability for matrix sparse factorization in the case of 2 factors has been studied in [15, Chapter 7], with a particular setting where the support of each factor is fixed. Then, in this setting, [15] focuses on conditions to identify the entries of the pair of factors. These conditions are based on the so-called *rank 1 contributions representation* for a pair of factors $(\boldsymbol{X}, \boldsymbol{Y}) \in \mathbb{C}^{n \times r} \times \mathbb{C}^{r \times m}$, which is the following tuple of matrices of rank at most 1:

$$(\boldsymbol{X}_{\bullet i} \boldsymbol{Y}_{i \bullet})_{i=1}^{r}, \tag{1.4}$$

where r is the number of columns in $\boldsymbol{X}$ (or the number of rows in $\boldsymbol{Y}$), $\boldsymbol{X}_{\bullet i}$ is the i-th column of $\boldsymbol{X}$, and $\boldsymbol{Y}_{i \bullet}$ is the i-th row of $\boldsymbol{Y}$. This representation has the important property that the matrix product $\boldsymbol{XY}$ can be written as the sum $\sum_{i=1}^{r} \boldsymbol{X}_{\bullet i} \boldsymbol{Y}_{i \bullet}$. This notion of rank 1 contributions representation will be further presented in Section 4.2. With this representation, it is possible to claim that, when the supports are fixed, the entries of the pair of factors $(\boldsymbol{X}, \boldsymbol{Y})$ are identifiable (up to equivalence relations) when its rank 1 contributions $\boldsymbol{X}_{\bullet i} \boldsymbol{Y}_{i \bullet}$ for $i \in [\![1, r]\!]$ are identifiable (up to equivalence relations). Using a combinatorial algebraic approach to complete rank 1 matrices like in [14], [15] establishes some conditions based on matrix completability to identify the rank 1 contributions. This representation leads to necessary conditions [15, Chapter 7, Lemma 3] and sufficient conditions [15, Chapter 7, Lemma 2] for identifying the entries of a pair of factors $(\boldsymbol{X}, \boldsymbol{Y})$ when their support is fixed. However, these conditions are not both necessary and sufficient. Chapter 4 will be based on these existing results, and will show that we can use some more precise matrix completability conditions to characterize identifiability of entries when the left and right support are fixed.

1.4 Contributions

The main contributions of this work are:

1. proving some necessary conditions for identifiability in matrix sparse factorization by considering two important problem variations: the case where one factor is fixed, and the case where the supports of the pair of factors are fixed;

2. a characterization of identifiability in the case where the left factor is fixed, using linear independence of specific subsets of columns in the left factor;

3. a characterization of identifiability in the case where the left and right supports are fixed, using the lifting principle and rank 1 matrix completability.

1.4.1 Necessary conditions of identifiability

Our framework for studying identifiability in matrix sparse factorization will be based on the notion of *family of allowed supports*, which is essentially a family of matrix supports. A so-called *allowed support* will then be a support belonging to this family, and we say that a matrix $\boldsymbol{M}$ is sparse if it has an allowed support. Similarly, we can consider the notion of *family of allowed pairs of supports*, which is a family of pairs of supports $(\boldsymbol{S_L}, \boldsymbol{S_R})$, where $\boldsymbol{S_L}$ is a support for left matrices, and $\boldsymbol{S_R}$ is a support for right matrices. We will further detail these notions in Section 2.1.2. Because of unavoidable scaling and permutation ambiguities in matrix product, we also consider equivalence relations between pairs of factors: $(\boldsymbol{X}, \boldsymbol{Y})$ and $(\boldsymbol{X'}, \boldsymbol{Y'})$ are equivalent if there exists a scaled permutation matrix $\boldsymbol{C} \in \mathcal{C}$ such that $\boldsymbol{XC} = \boldsymbol{X'}$ and $\boldsymbol{C}^{-1}\boldsymbol{Y} = \boldsymbol{Y'}$, where $\mathcal{C}$ is the group of scaled permutation matrix defined at (1.3). Then, we can define identifiability of a pair of factor $(\boldsymbol{X}, \boldsymbol{Y})$

given a family of allowed pairs of supports, in the sense that it is the unique pair of factors with an allowed support for the observed matrix $\boldsymbol{Z} := \boldsymbol{X}\boldsymbol{Y}$, up to scaling and permutation equivalence. We then analyze this notion of identifiability by considering two specific problem variations.

1. In a first problem variation, one of the two factors, the left one, is fixed, and we want to identify the right factor $\boldsymbol{Y}$. We show that identifiability of the right factor when fixing the left factor, called *right identifiability*, is a necessary condition for identifiability in the generic case.

2. In a second problem variation, the family of allowed pairs of supports is reduced to a singleton, meaning that the supports of the left and right factor are fixed. Identifiability of the pair of factors when fixing their support will be called *fixed-support identifiability*. Since a singleton is a particular instance for a family of allowed pairs of supports, fixed-support identifiability is a necessary condition for identifiability in the generic case.

1.4.2 Conditions for right identifiability

When fixing the left factor $\boldsymbol{X}$, identifying the right factor becomes a linear inverse problem with sparsity constraints derived from the ones of the original bilinear inverse problem. Indeed, given a fixed left factor $\boldsymbol{X}$, the mapping $\boldsymbol{Y} \mapsto \boldsymbol{X}\boldsymbol{Y}$ is linear. In the specific case where the fixed left factor $\boldsymbol{X}$ has a non-degenerate structure, it is possible to remove scaling and permutation ambiguity, and we show that right identifiability in this case is equivalent to the injectivity of $\boldsymbol{Y} \mapsto \boldsymbol{X}\boldsymbol{Y}$ restricted to the set of right factors $\boldsymbol{Y}$ which have an allowed support Then, for particular families of allowed right supports, we can prove in Theorem 3.1 some easily verifiable conditions for such injectivity, because they can be expressed as the linear independence of specific subsets of the left factor's columns. For instance, when considering the family of right supports which are k-sparse by column (no more than k nonzero entries per column), the right factor is identifiable if, and only if, every subsets of $2k$ columns of the left factor is linearly independent. This is actually the same necessary and sufficient condition for the injectivity of $\boldsymbol{y} \mapsto \boldsymbol{X}\boldsymbol{y}$ defined on the set of vectors which are k-sparse (at most k nonzero entries), as claimed by [12, Theorem 2.13] in compressive sensing and sparse recovery literature. Therefore we can see Theorem 3.1 and its extension Theorem A.1 as a generalization of [12, Theorem 2.13], where instead of considering a specific sparsity model (k-sparse vectors), we consider any structured sparsity model, in the spirit of model-based compressive sensing presented in Section 1.3.1. The structure of these sparsity model will be given by the choice of the family of allowed (pairs of) supports.

1.4.3 Conditions for fixed-support identifiability

When fixing a pair of supports, we can use the *lifting principle* to study identifiability of the entries. Essentially, we show in Proposition 4.2 that the lifting principle characterizes fixed-support identifiability using triviality of the null space of a linear operator $\mathscr{S}$ defined on a restricted set of matrices which are at most of rank 2, and intersected with a secent set determined by the fixed pair of supports. The lifting principle implicitly uses the *rank 1 contributions* representation of a pair of factors $(\boldsymbol{X}, \boldsymbol{Y})$, defined by (1.4). Then, it is natural to use *rank 1 matrix completability* to characterize the triviality of the restricted rank 2 null space of the lifting operator $\mathscr{S}$. Rank 1 matrix completability is the possibility of filling the missing values of a rank 1 matrix by only observing a subset of its entries. Then, the idea is that a pair of factors $(\boldsymbol{X}, \boldsymbol{Y})$ has identifiable entries if it is possible to complete one by one its rank 1 contributions $(\boldsymbol{X}_{\bullet i}\boldsymbol{Y}_{i\bullet})$ only from its *observable entries*, which are entries with an index only belonging to the support of the i-th rank 1 contribution. This idea is illustrated in Figure 1.1. In Chapter 4, we will define the notion of *iterative completability from observable supports*, and Proposition 4.3 shows that it is a sufficient condition for fixed-support identifiability. Then, Proposition 4.4 shows that iterative completability is a necessary condition for fixed-support identifiability, in the specific case of $r = 2$, but not for $r > 2$, where r is the number of columns (resp. rows) for the left (resp. right) factor.

$$
\boldsymbol{Z} = \begin{pmatrix} 0 & \mathbf{1} & 0 \\ \mathbf{2} & \mathbf{3} & \mathbf{3} \\ \mathbf{2} & \mathbf{1} & \mathbf{3} \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{1} & 0 \\ 0 & ? & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ \mathbf{2} & ? & \mathbf{3} \\ \mathbf{2} & \mathbf{1} & \mathbf{3} \end{pmatrix}
$$

$$
= \begin{pmatrix} 0 & \mathbf{1} & 0 \\ 0 & ? & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ \mathbf{2} & \mathbf{1} & \mathbf{3} \\ \mathbf{2} & \mathbf{1} & \mathbf{3} \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{1} & 0 \\ 0 & \mathbf{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ \mathbf{2} & \mathbf{1} & \mathbf{3} \\ \mathbf{2} & \mathbf{1} & \mathbf{3} \end{pmatrix}
$$

Figure 1.1: Given the matrix $\boldsymbol{Z}$, is there a unique pair of factors $(\boldsymbol{X}, \boldsymbol{Y})$, up to equivalence relations verifying $\boldsymbol{XY} = \boldsymbol{Z}$, such that the left support is $\begin{pmatrix} \star & 0 \\ \star & \star \\ 0 & \star \end{pmatrix}$ and the right support is $\begin{pmatrix} 0 & \star & 0 \\ \star & \star & \star \end{pmatrix}$, where the symbol $(\star)$ denotes a nonzero entry? Yes, because we can first identify the rank 1 contribution in green from its observable entries (in bold), which are entries that are not covered by other rank 1 contributions. Non-observable entries are marked by (?), but they can be completed from observable entries by using the rank 1 constraint. Then, having identifying the rank 1 contribution in green, we can identify the rank 1 contribution in red. The property of completing iteratively the green rank 1 contribution and the red rank 1 contribution will be called in Chapter 4 *iterative completability from observable supports*, which can characterize fixed-support identifiability.

1.5 Reading guidelines

The following report will be organized in the following way:

- Chapter 2 introduces our framework for studying identifiability in matrix sparse factorization with important notations, and present the necessary conditions for identifiability derived from the analysis of the two considered problem variations;
- Chapter 3 addresses right identifiability, which is the variation of the identifiability problem where the left factor is fixed;
- Chapter 4 addresses fixed-support identifiability, which is the variation of the identifiability problem where the pair of supports is fixed;
- Chapter 5 will conclude this work by establishing some parallels between right identifiability results and fixed-support identifiability results.

Important extensions As we want to present the most important ideas of this work in a limited number of pages, we limit the length of the main text to approximately 30 pages. However, this report also contains important extensions suggested by the main text, which are deferred to the appendices. We refer the reader to these extensions in Appendix A for a more advanced lecture of this report.

Proofs For more fluidity in the lecture, we also sometimes defer proofs to the appendices at the end of this report in Appendix B. Lemmas, propositions, theorems or corollaries marked by a star like $(\star)$ will have a proof deferred to the appendices, and the reader can click on the result's number in red to jump to the page of the proof in the appendices. In the appendices, the result we want to prove is repeated for easier reading of the proof, and the reader can click again on the result's number in red to jump back to the main text. Under some results, we also provide in the main text a sketch of proof which can gives the main idea of the proof deferred to the appendices.

Chapter 2

Necessary conditions of identifiability

In this chapter, we introduce the notion of *family of allowed supports*, and the one of scaling and permutation equivalence between pairs of factors or pairs of supports. This allows for the definition of *generic identifiability*, *right identifiability* and *fixed-support identifiability*. A summary of different notions of identifiability introduced in this work will be given in Section 2.4.

2.1 Framework for studying identifiability

2.1.1 Basic notations

Symbol font Vectors will be denoted in bold lowercase letters, like the vector $\boldsymbol{v}$. Matrices will be denoted in bold capital letters, like $\boldsymbol{M}$. Sets of matrices will be denoted with calligraphic capital letters, like $\mathcal{R}_k$. Linear operators will be denoted with script capital letters, like $\mathscr{S}$.

Set, pair, tuple For any integer $p \in \mathbb{N}^*$, the set $[\![p]\!]$ is the set of integers $\{1, ..., p\}$, and for any integers $p, q \in \mathbb{N}^*$ such that $p \leq q$, the set $[\![p; q]\!]$ is the set of integers $\{p, p+1, ..., q-1, q\}$. Tuple of elements will be underlined, like the tuple of matrices $\underline{\boldsymbol{M}}$. Then, the i-th element of $\underline{\boldsymbol{M}}$ will be denoted $\boldsymbol{M_i}$, without the underline for a lighter notation. Pairs of supports (the notion of support will be precised in Section 2.1.2), will be denoted with a hat, like $\hat{\boldsymbol{S}}$. For $\mathcal{A}$ a set of objects, we write $A \subseteq \mathcal{A}$ when A is a subset of $\mathcal{A}$, $A \subsetneq \mathcal{A}$ when A is a subset of $\mathcal{A}$ different from $\mathcal{A}$, and $\mathcal{P}(\mathcal{A})$ is the set of all subsets of $\mathcal{A}$. For a finite set A, the cardinal of A is written $\text{card}(A)$ or $|A|$. The complementary set of $A \subseteq \mathcal{A}$ in $\mathcal{A}$ is $\mathcal{A} \backslash A$ or A^c when there is no ambiguity. For A, B two sets in a linear space, we denote the difference set $A - B := \{a - b \mid (a, b) \in A \times B\}$. For two sets A, B, we will use the notation $A \Delta B := (A \backslash B) \cup (B \backslash A)$ for the symmetric difference.

Boolean domain We will use the notation $\mathbb{B} = \{0, 1\}$.

Vector Let $p \in \mathbb{N}^*$. The set of complex vectors of size p is denoted $\mathbb{C}^p$. The entry indexed by $i \in [\![p]\!]$ for a vector $\boldsymbol{v} \in \mathbb{C}^p$ is denoted $\boldsymbol{v}_i \in \mathbb{C}$. For $i \in [\![p]\!]$, the vector $\boldsymbol{e_i} \in \mathbb{C}^p$ is the i-th vector of the canonical basis in $\mathbb{C}^p$, which is full of zeros except at the i-th coordinate where the entry is 1. The ℓ_0-norm $\| \cdot \|_0$ is the number of nonzero elements in the vector. For a subspace $F \subseteq \mathbb{C}^p$, we denote $F^\perp$ the orthogonal space of F, for the canonical inner product in $\mathbb{C}^p$.

Matrix Let $p, q \in \mathbb{N}^*$. The set of complex matrices of size $p \times q$ is denoted $\mathbb{C}^{p \times q}$. The entry indexed by $(i, j) \in [\![p]\!] \times [\![q]\!]$ for a matrix $\boldsymbol{M}$ is denoted $\boldsymbol{M}_{ij} \in \mathbb{C}$. For $(i, j) \in [\![p]\!] \times [\![q]\!]$, the matrix $\boldsymbol{E_{ij}} \in \mathbb{C}^{p \times q}$ is the element (i, j) of the canonical basis in $\mathbb{C}^{p \times q}$, which is full of zeros except at the index (i,j) where the entry is 1. For any integer k, we denote $\mathcal{R}_k(p,q)$ the set of matrices in $\mathbb{C}^{p\times q}$ with a rank at most k, and the notation (p,q) is omitted when there is no ambiguity. The identity matrix of size p is denoted $\boldsymbol{I_p} \in \mathbb{C}^{p\times p}$. The symbol $\otimes$ is the Kronecker product. Let $\boldsymbol{M} \in \mathbb{C}^{p\times q}$. We denote $\boldsymbol{M}_{\bullet j} := \boldsymbol{M}\boldsymbol{e_j}$ the j-th column of $\boldsymbol{M}$ for $j \in [\![q]\!]$, and $\boldsymbol{M_{i\bullet}} = \boldsymbol{e_i^T}\boldsymbol{M}$ the i-th row of $\boldsymbol{M}$ for $i \in [\![p]\!]$. Given a set of indices $S \subseteq [\![p]\!] \times [\![q]\!]$ and a matrix $\boldsymbol{M} \in \mathbb{C}^{p\times q}$, we denote the restriction of the matrix $\boldsymbol{M}$ on S as $\boldsymbol{M}_{|S} := (\boldsymbol{M}_{ij})_{(i,j)\in S} \in \mathbb{C}^S$, which is a family of complex scalars indexed by S. The image of $\boldsymbol{M}$ will be denoted $\mathrm{Im}(\boldsymbol{M}) := \mathrm{span}((\boldsymbol{M}\boldsymbol{e_j})_{j\in[\![q]\!]}) \subseteq \mathbb{C}^n$. The kernel of $\boldsymbol{M}$ will be denoted $\mathrm{Ker}(\boldsymbol{M}) := \{\boldsymbol{v} \in \mathbb{C}^m \mid \boldsymbol{M}\boldsymbol{v} = 0\}$. The ℓ_0-norm $\|\cdot\|_0$ is the number of nonzero entries in the matrix.

2.1.2 Support, family of allowed supports

We introduce the notion of *family of allowed supports*, and say that a matrix is sparse if its support belongs to this family. This framework allows us to consider any kind of sparsity structure, depending on the chosen family of supports. For instance, one can consider the family of supports with no more than k nonzero entries per column, or the family of supports with no more than l nonzero entries per row, etc.

Support of a vector, matrix To clarify the notion of family of allowed supports, we start by defining the support of a matrix. In this work, we represent a matrix support of size $p \times q$ (resp. a vector support of size p) as a binary matrix of size $p \times q$ (resp. a binary vector of size p), where the 1 entries in the binary matrix (resp. vector) corresponds to nonzero values in the matrix (resp. vector).

In other words, for any matrix $\boldsymbol{M} \in \mathbb{C}^{p\times q}$, the support of $\boldsymbol{M}$ denoted $\mathrm{supp}(\boldsymbol{M}) \in \mathbb{B}^{p\times q}$ is the binary matrix defined by:

$$\forall (i,j) \in [\![p]\!] \times]\!]q]\!], \quad \mathrm{supp}(\boldsymbol{M})_{ij} = 1 \iff \boldsymbol{M}_{ij} \neq 0. \tag{2.1}$$

Similarly, for any vector $\boldsymbol{v} \in \mathbb{C}^p$, the support of $\boldsymbol{v}$ denoted $\mathrm{supp}(\boldsymbol{v}) \in \mathbb{B}^p$ is the binary vector defined by:

$$\forall i \in [\![p]\!], \quad \mathrm{supp}(\boldsymbol{v})_i = 1 \iff \boldsymbol{v}_i \neq 0. \tag{2.2}$$

Traditionally, matrix (resp. vector) support is defined as a subset of indices for which the entry is nonzero. Here, we represent it as a binary matrix (resp. vector), because it keeps the information about the size of the matrix (resp. vector).

Binary matrix, vector as a subset of indices However, it might happen that we want to manipulate the subset definition of a support. For any binary matrix $\boldsymbol{S} \in \mathbb{B}^{p\times q}$, we might want to see it as the subset of indices:

$$\{(i,j) \in [\![p]\!] \times [\![q]\!] \mid \boldsymbol{S}_{ij} = 1\}, \tag{2.3}$$

and similarly for any binary vector $\boldsymbol{s} \in \mathbb{B}^p$, we might want to see it as the subset of indices:

$$\{i \in [\![p]\!] \mid \boldsymbol{s}_i = 1\}. \tag{2.4}$$

Therefore, in this work, we will use the following abuse of notation. Whenever binary matrices or binary vectors are manipulated with other sets, or with set operations like ($\in$, $\subseteq$, $\cup$, $\cap$, $\setminus$, etc.), the reader should see these binary matrices or binary vectors as the set of indices where the entries are nonzero, given by (2.3) or (2.4). For instance, we will usually write things like:

- (mixing binary matrices and sets) $\boldsymbol{S} = [\![p]\!] \times [\![q]\!]$ for $\boldsymbol{S} \in \mathbb{B}^{p\times q}$
- (membership) index $(i,j) \in \boldsymbol{S}$ for $\boldsymbol{S} \in \mathbb{B}^{p\times q}$;
- (inclusion) $\boldsymbol{S'} \subseteq \boldsymbol{S}$ for $\boldsymbol{S'}, \boldsymbol{S} \in \mathbb{B}^{p\times q}$;
- (union, intersection) $\mathrm{supp}(\boldsymbol{A}) \cup \mathrm{supp}(\boldsymbol{B})$, or $\mathrm{supp}(\boldsymbol{A}) \cap \mathrm{supp}(\boldsymbol{B})$ for $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{C}^{p\times q}$;
- (complementary) $\boldsymbol{S}\setminus\boldsymbol{S'}$ for $\boldsymbol{S'}, \boldsymbol{S} \in \mathbb{B}^{p\times q}$.

Family of allowed supports Now, a family of allowed supports will be a subset of supports, represented by binary matrices. For example, when considering matrices of size $p \times q$, a family of allowed supports $\Omega \subseteq \mathbb{B}^{p\times q}$ is a subset of $\mathbb{B}^{p\times q}$.

Example 2.1 (Family of allowed supports)**.** We illustrate some families of allowed supports that will be typically considered for application.

- Allowing only one specific support: $\Omega = \{\boldsymbol{S}\}$ for a specific $\boldsymbol{S} \in \mathbb{B}^{p\times q}$.
- Allowing a given global sparsity: $\Omega = \{\boldsymbol{S} \in \mathbb{B}^{p\times q} \mid \|\boldsymbol{S}\|_0 \leq s\}$, for a given parameter $s \in [\![pq]\!]$.
- Allowing a given sparsity by row: for a given parameter $k \in [\![q]\!]$, $\Omega = \{\boldsymbol{S} \in \mathbb{B}^{p\times q} \mid \|\boldsymbol{S}_{i\bullet}\|_0 \leq k,\ \forall i \in [\![p]\!]\}$.
- Allowing a given sparsity by column: for a given parameter $l \in [\![p]\!]$, $\Omega = \{\boldsymbol{S} \in \mathbb{B}^{p\times q} \mid \|\boldsymbol{S}_{\bullet j}\|_0 \leq l,\ \forall j \in [\![q]\!]\}$.
- Allowing a given sparsity by row and by column: for given parameters $(k, l) \in [\![p]\!] \times [\![q]\!]$, $\Omega = \{\boldsymbol{S} \in \mathbb{B}^{p\times q} \mid \|\boldsymbol{S}_{i\bullet}\|_0 \leq k \text{ and } \|\boldsymbol{S}_{\bullet j}\|_0 \leq l,\ \forall (i,j) \in [\![p]\!] \times [\![q]\!]\}$.

Matrices supported by a support Let $\boldsymbol{S} \in \mathbb{B}^{p\times q}$ be a matrix support. We define $\Sigma_{\boldsymbol{S}}$ as the set of matrices exactly supported by $\boldsymbol{S}$, which is the set:

$$\Sigma_{\boldsymbol{S}} := \{\boldsymbol{M} \in \mathbb{C}^{p\times q} \mid \operatorname{supp}(\boldsymbol{M}) = \boldsymbol{S}\}. \tag{2.5}$$

We remark that $\Sigma_{\boldsymbol{S}}$ is not a linear space, because we require the equality $\operatorname{supp}(\boldsymbol{M}) = S$. Indeed, matrices in $\Sigma_{\boldsymbol{S}}$ has nonzero values on $\boldsymbol{S}$. Therefore we also define $\overline{\Sigma}_{\boldsymbol{S}}$ the set of matrices supported by $\boldsymbol{S}$ in a large sense, which is the set:

$$\overline{\Sigma}_{\boldsymbol{S}} := \{\boldsymbol{M} \in \mathbb{C}^{p\times q} \mid \operatorname{supp}(\boldsymbol{M}) \subseteq \boldsymbol{S}\}. \tag{2.6}$$

Then, $\overline{\Sigma}_{\boldsymbol{S}}$ becomes a linear space, because it allows zero values on $\boldsymbol{S}$.

Vectors supported by a support We will use similar notations for vectors supported by a support. Let $\boldsymbol{s} \in \mathbb{B}^p$ be a vector support. We define:

$$\Sigma_{\boldsymbol{s}} := \{\boldsymbol{v} \in \mathbb{C}^p \mid \operatorname{supp}(\boldsymbol{v}) = \boldsymbol{s}\}, \tag{2.7}$$

$$\overline{\Sigma}_{\boldsymbol{s}} := \{\boldsymbol{v} \in \mathbb{C}^p \mid \operatorname{supp}(\boldsymbol{v}) \subseteq \boldsymbol{s}\}. \tag{2.8}$$

Set of matrices with an allowed support Let $\Omega \subseteq \mathbb{B}^{p\times q}$ be a family of allowed supports. Then, based on the notation given by (2.5), we define the set of matrices with an allowed support for the family Ω as:

$$\Sigma_\Omega := \bigcup_{\boldsymbol{S}\in\Omega} \Sigma_{\boldsymbol{S}} = \{\boldsymbol{M} \in \mathbb{C}^{p\times q} \mid (\exists \boldsymbol{S} \in \Omega, \operatorname{supp}(\boldsymbol{M}) = \boldsymbol{S})\}. \tag{2.9}$$

In other words, a matrix $\boldsymbol{M}$ has an allowed support if $\boldsymbol{M} \in \Sigma_\Omega$. Since we also want to consider the set of matrices with a support included in an allowed support, we also define, based on the notation given by (2.6), the set:

$$\overline{\Sigma}_\Omega := \bigcup_{\boldsymbol{S}\in\Omega} \overline{\Sigma}_{\boldsymbol{S}} = \{\boldsymbol{M} \in \mathbb{C}^{p\times q} \mid (\exists \boldsymbol{S} \in \Omega, \operatorname{supp}(\boldsymbol{M}) \subseteq \boldsymbol{S})\}. \tag{2.10}$$

Define now the *closure* of Ω, denoted $\overline{\Omega}$, as:

$$\overline{\Omega} := \bigcup_{\boldsymbol{S}\in\Omega} \{\boldsymbol{S'} \in \mathbb{B}^{p\times q} \mid \boldsymbol{S'} \subseteq \boldsymbol{S}\}. \tag{2.11}$$

Then, it is in fact possible to formulate the set $\overline{\Sigma}_\Omega$ as the set of matrices with an allowed support for the closed family $\overline{\Omega}$:

$$\overline{\Sigma}_\Omega = \Sigma_{\overline{\Omega}}. \tag{2.12}$$

In other words, by considering the closed family $\overline{\Omega}$ of allowed supports, each support $\boldsymbol{S'}$ which is included in a support $\boldsymbol{S} \in \Omega$ becomes an allowed support.

Model sets, secant sets As we can see, the overline on a set will be used when considering support inclusion instead of support equality, in the spirit of (2.5) and (2.6). Using the terminology from [5], a set $\Sigma_\bullet$ without overline will be usually called *non-extended model set*, in contrast to a set $\overline{\Sigma}_\bullet$ with an overline which will be called *extended model set*. A *secant set* is then defined as the difference between two model sets. For instance, for two given supports $\boldsymbol{S}, \boldsymbol{S'} \in \mathbb{B}^{p\times q}$, the difference $\Sigma_{\boldsymbol{S}} - \Sigma_{\boldsymbol{S'}}$ is a secant set.

2.1.3 Scaling and permutation equivalence

Because of the nature of the matrix product, it is necessary to consider some natural equivalence relations between pairs of factors. Indeed, instead of talking about uniqueness of a pair of sparse factors $(\boldsymbol{X}, \boldsymbol{Y})$ in the factorization of the observed matrix $\boldsymbol{Z} := \boldsymbol{XY}$, we talk about identifiability of the pair $(\boldsymbol{X}, \boldsymbol{Y})$, in the sense that the observed matrix $\boldsymbol{Z} := \boldsymbol{XY}$ admits a unique sparse factorization up to scaling and permutation equivalence.

1. On the one hand, scaling ambiguity is inherent in bilinear inverse problems [9]: for any linear space E and F, for any bilinear operator $\boldsymbol{B} : E \times F \to \mathbb{C}$ and for any $(\boldsymbol{x}, \boldsymbol{y}) \in E \times F$, we have $\boldsymbol{B}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{B}(\lambda \boldsymbol{x}, \frac{1}{\lambda}\boldsymbol{y})$ for all $\lambda \in \mathbb{C}\backslash\{0\}$. In particular, the matrix product $\mathbb{C}^{n\times r} \times \mathbb{C}^{r\times m} \to \mathbb{C}^{n\times m}$, $(\boldsymbol{X}, \boldsymbol{Y}) \mapsto \boldsymbol{XY}$ is a bilinear operator.

2. On the other hand, this operator has also an invariance of permutation because of the sum operator in the matrix product. For $(\boldsymbol{X}, \boldsymbol{Y}) \in \mathbb{C}^{n\times r} \times \mathbb{C}^{r\times m}$, we have:

$$\boldsymbol{XY} = \sum_{i=1}^{r} \boldsymbol{X}_{\bullet\sigma(i)} \boldsymbol{Y}_{\sigma(i)\bullet} \tag{2.13}$$

   for all permutations $\sigma : [\![r]\!] \to [\![r]\!]$.

Equivalent pairs of factors It is then natural to define below the notion of *scaled permutation matrix*, which is a matrix product between a diagonal matrix with nonzero diagonal entries, and a permutation matrix. The diagonal matrix represents scaling equivalence, while the permutation matrix represents permutation equivalence.

Definition 2.1 (Scaled permutation matrix)**.** A scaled permutation matrix $\boldsymbol{C} \in \mathbb{C}^{r\times r}$ is an invertible matrix which can be written as a product $\boldsymbol{C} = \boldsymbol{DP}$ where $\boldsymbol{D} \in \mathbb{C}^{r\times r}$ is a diagonal matrix with nonzero entries on the diagonal, and $\boldsymbol{P} \in \mathbb{B}^{r\times r}$ a permutation matrix.

Let $(\boldsymbol{X}, \boldsymbol{Y}), (\boldsymbol{X'}, \boldsymbol{Y'}) \in \mathbb{C}^{n\times r} \times \mathbb{C}^{r\times m}$ be two pairs of factors. We say that $(\boldsymbol{X}, \boldsymbol{Y})$ and $(\boldsymbol{X'}, \boldsymbol{Y'})$ are equivalent in the sense of scaling and permutation equivalence, and write:

$$(\boldsymbol{X}, \boldsymbol{Y}) \sim (\boldsymbol{X'}, \boldsymbol{Y'}), \tag{2.14}$$

if there exists a scaled permutation matrix $\boldsymbol{C} \in \mathbb{C}^{r\times r}$ such that $\boldsymbol{XC} = \boldsymbol{X'}$ and $\boldsymbol{C}^{-1}\boldsymbol{Y} = \boldsymbol{Y'}$. The class of pairs equivalent to a given pair $(\boldsymbol{X}, \boldsymbol{Y})$ in the sense of permutation and scaling equivalence is denoted $[\boldsymbol{X}, \boldsymbol{Y}]$. We say that $(\boldsymbol{X}, \boldsymbol{Y})$ and $(\boldsymbol{X}, \boldsymbol{Y})$ are equivalent in the sense of scaling equivalence only, and write:

$$(\boldsymbol{X}, \boldsymbol{Y}) \sim_s (\boldsymbol{X'}, \boldsymbol{Y'}), \tag{2.15}$$

if there exists a diagonal matrix $\boldsymbol{D} \in \mathbb{C}^{r\times r}$ with nonzero diagonal element such that $\boldsymbol{XD} = \boldsymbol{X'}$ and $\boldsymbol{D}^{-1}\boldsymbol{Y} = \boldsymbol{Y'}$. The class of pairs equivalent to a given pair $(\boldsymbol{X}, \boldsymbol{Y})$ in the sense of scaling equivalence only is denoted $[\boldsymbol{X}, \boldsymbol{Y}]_s$.

Pair of supports Since we consider pairs of factors in matrix sparse factorization, it is also natural to consider pairs of supports, which is just a pair of binary matrices representing the left and the right support. We will denote such elements by $\hat{\boldsymbol{S}} \in \mathbb{B}^{n \times r} \times \mathbb{B}^{r \times m}$. For any $\hat{\boldsymbol{S}} \in \mathbb{B}^{n \times r} \times \mathbb{B}^{r \times m}$, we will usually write:

$$\hat{\boldsymbol{S}} = (\boldsymbol{S_L}, \boldsymbol{S_R}), \tag{2.16}$$

where $\boldsymbol{S_L}$ is the left factor and $\boldsymbol{S_R}$ is the right factor. In the following we will use also the following notations for the model sets given by a pair of supports $\hat{\boldsymbol{S}} \in \mathbb{B}^{n \times r} \times \mathbb{B}^{r \times m}$:

$$\Sigma_{\hat{\boldsymbol{S}}} := \Sigma_{\boldsymbol{S_L}} \times \Sigma_{\boldsymbol{S_R}}, \tag{2.17}$$

$$\overline{\Sigma}_{\hat{\boldsymbol{S}}} := \overline{\Sigma}_{\boldsymbol{S_L}} \times \overline{\Sigma}_{\boldsymbol{S_R}}. \tag{2.18}$$

Equivalent pairs of supports Then, we can also define an equivalence relation for pairs of supports. In fact, we only need to consider permutation equivalence for pairs of supports, since scaling ambiguity does not exist for pairs of supports. Let $\hat{\boldsymbol{S}}, \hat{\boldsymbol{S}}' \in \mathbb{B}^{n \times r} \times \mathbb{B}^{r \times m}$ be two pairs of supports. We say that $\hat{\boldsymbol{S}}, \hat{\boldsymbol{S}}'$ are equivalent, and write:

$$\hat{\boldsymbol{S}} \sim \hat{\boldsymbol{S}}', \tag{2.19}$$

if there exists a permutation matrix $\boldsymbol{P} \in \mathbb{B}^{r \times r}$ such that $\boldsymbol{S_L P} = \boldsymbol{S'_L}$ and $\boldsymbol{P}^T \boldsymbol{S_R} = \boldsymbol{S'_R}$. The class of pairs equivalent to a given pair $\hat{\boldsymbol{S}}$ is denoted $[\hat{\boldsymbol{S}}]$.

Family of allowed pairs of supports Therefore, a family $\hat{\Omega} \subseteq \mathbb{B}^{n \times r} \times \mathbb{B}^{r \times m}$ of allowed pairs of supports is a subset of $\mathbb{B}^{n \times r} \times \mathbb{B}^{r \times m}$. Given the introduced permutation equivalence relation for pairs of supports, we can define below the notion of *stability by permutation* for a family of allowed pairs of supports.

Definition 2.2 (Family of allowed pairs of supports stable by permutation)**.** Let $\hat{\Omega} \subseteq \mathbb{B}^{n \times r} \times \mathbb{B}^{r \times m}$ be a family of allowed pairs of supports. We say that $\hat{\Omega}$ is stable by permutation if for all $\hat{\boldsymbol{S}} \in \hat{\Omega}$, we have $[\hat{\boldsymbol{S}}] \subseteq \hat{\Omega}$.

Example 2.2 (Family of allowed pairs of supports)**.** We illustrate some families of allowed pairs of supports that will be typically considered for application.

- Allowing only one equivalence class of pairs of supports: $\hat{\Omega} = [\hat{\boldsymbol{S}}]$ for a specific $\hat{\boldsymbol{S}} \in \mathbb{B}^{n \times r} \times \mathbb{B}^{r \times m}$.
- Allowing a given sparsity by column for the left factor and a given sparsity by row for the right factor [15, 22]: for given parameters $(k, l) \in [\![n]\!] \times [\![m]\!]$, $\hat{\Omega} = \{(\boldsymbol{S_L}, \boldsymbol{S_R}) \in \mathbb{B}^{n \times r} \times \mathbb{B}^{r \times m} \mid \|(\boldsymbol{S_L})_{\bullet i}\|_0 \leq k, \|(\boldsymbol{S_R})_{i \bullet}\|_0 \leq l, \forall i \in [\![r]\!]\}$.

Since most examples of families that will be considered for application have such property, we will assume from now on that any family of pairs of supports $\hat{\Omega}$ is always stable by permutation.

Set of pairs of factors with an allowed support Let $\hat{\Omega} \subseteq \mathbb{B}^{n \times r} \times \mathbb{B}^{r \times m}$ be a family of allowed pairs of support. Similarly to the previous section, we can define the set of pairs of factors that have an allowed support for the family $\hat{\Omega}$ as:

$$\Sigma_{\hat{\Omega}} := \bigcup_{\hat{\boldsymbol{S}} \in \hat{\Omega}} \Sigma_{\hat{\boldsymbol{S}}}. \tag{2.20}$$

In the same spirit of (2.11) and (2.12), we can define the closure of $\hat{\Omega}$, denoted $\overline{\hat{\Omega}}$, as:

$$\overline{\hat{\Omega}} := \bigcup_{\hat{\boldsymbol{S}} \in \hat{\Omega}} \{\hat{\boldsymbol{S}}' \in \mathbb{B}^{n \times r} \times \mathbb{B}^{r \times m} \mid \boldsymbol{S'_L} \subseteq \boldsymbol{S_L} \text{ and } \boldsymbol{S'_R} \subseteq \boldsymbol{S_R}\}, \tag{2.21}$$

which leads to the equality and the notation:

$$\Sigma_{\overline{\hat{\Omega}}} = \bigcup_{\hat{\boldsymbol{S}} \in \hat{\Omega}} \overline{\Sigma}_{\hat{\boldsymbol{S}}} := \overline{\Sigma}_{\hat{\Omega}}. \tag{2.22}$$

2.2 Problem formulation: instance and global identifiability

Based on the framework introduced in Section 2.1, we can address the issue of identifiability in matrix sparse factorization, which is essentially: "does there exist a unique sparse factorization for a given observed matrix, up to natural equivalence relations?" Let $\hat{\Omega} \subseteq \mathbb{B}^{n\times r} \times \mathbb{B}^{r\times m}$ be a family of allowed pairs of supports, and $(\boldsymbol{X}, \boldsymbol{Y}) \in \Sigma_{\hat{\Omega}}$ be a pair of factors of reference with an allowed support. Denote the observed matrix product $\boldsymbol{Z} := \boldsymbol{XY}$. We can then define the following bilinear inverse problem (2.23):

$$\begin{aligned} \text{find} \quad & (\boldsymbol{A}, \boldsymbol{B}) \\ \text{subject to} \quad & \boldsymbol{AB} = \boldsymbol{Z}, \\ & (\boldsymbol{A}, \boldsymbol{B}) \in \Sigma_{\hat{\Omega}}. \end{aligned} \tag{2.23}$$

Remark. One can also consider the particular case where $\hat{\Omega}$ is a closed family of allowed pairs of factors, in the sense that $\overline{\hat{\Omega}} = \hat{\Omega}$. In this case, we can reformulate the bilinear inverse problem (2.23) by replacing $\Sigma_{\hat{\Omega}}$ with $\overline{\Sigma}_{\hat{\Omega}}$, since we have the equality (2.22).

By definition, $(\boldsymbol{X}, \boldsymbol{Y})$ is a solution of (2.23). But because of scaling and permutation ambiguities inherent in matrix product (see Section 2.1.3), $(\boldsymbol{X'}, \boldsymbol{Y'})$ is also a solution of (2.23) for all $(\boldsymbol{X'}, \boldsymbol{Y'}) \in [\boldsymbol{X}, \boldsymbol{Y}]$. We can then formulate the following definition of *generic identifiability* in matrix sparse factorization.

Definition 2.3 (Instance generic identifiability)**.** Let $\hat{\Omega} \subseteq \mathbb{B}^{n\times r} \times \mathbb{B}^{r\times m}$ be a family of allowed pairs of supports, and $(\boldsymbol{X}, \boldsymbol{Y}) \in \Sigma_{\hat{\Omega}}$ be a pair of factors. We say that $(\boldsymbol{X}, \boldsymbol{Y})$ is generically identifiable for the family $\hat{\Omega}$ if, for any $(\boldsymbol{A}, \boldsymbol{B}) \in \Sigma_{\hat{\Omega}}$ verifying $\boldsymbol{XY} = \boldsymbol{AB}$, we have $(\boldsymbol{X}, \boldsymbol{Y}) \sim (\boldsymbol{A}, \boldsymbol{B})$.

This definition of identifiability focuses on a specific instance of pair of factors $(\boldsymbol{X}, \boldsymbol{Y})$. In the case where we are only interested in conditions of identifiability which do not depend on a specific instance of pair of factors, it is natural to define a notion of *global generic identifiability.*

Definition 2.4 (Global generic identifiability)**.** Let $\hat{\Omega} \subseteq \mathbb{B}^{n\times r} \times \mathbb{B}^{r\times m}$ be a family of allowed pairs of supports. We say that $\hat{\Omega}$ is globally and generically identifiable if $(\boldsymbol{X}, \boldsymbol{Y})$ is generically identifiable for the family $\hat{\Omega}$ (Definition 2.3), for all pairs of factors $(\boldsymbol{X}, \boldsymbol{Y}) \in \Sigma_{\hat{\Omega}}$.

Therefore, given a pair of factors $(\boldsymbol{X}, \boldsymbol{Y})$ and a family of allowed pairs of supports $\hat{\Omega}$, conditions of global generic identifiability of $\hat{\Omega}$ are stronger than conditions of generic identifiability of $(\boldsymbol{X}, \boldsymbol{Y})$ for the family $\hat{\Omega}$. For the rest of the report, we will give some partial solutions to the following problem.

Problem 1. Let $\hat{\Omega} \subseteq \mathbb{B}^{n\times r} \times \mathbb{B}^{r\times m}$ be a family of allowed pairs of supports.

1. Under which conditions on $\hat{\Omega}$ do we have global generic identifiability of the family $\hat{\Omega}$ (Definition 2.4)?
2. We fix $(\boldsymbol{X}, \boldsymbol{Y}) \in \mathbb{C}^{n\times r} \times \mathbb{C}^{r\times m}$ a pair of factors with an allowed support. Under which conditions on $(\boldsymbol{X}, \boldsymbol{Y})$ and $\hat{\Omega}$ do we have generic identifiability of $(\boldsymbol{X}, \boldsymbol{Y})$ for the family $\hat{\Omega}$ (Definition 2.3)?

2.3 Analyzing the notion of identifiability

Precisely, the objective in Problem 1 is to find necessary and sufficient conditions for instance and global generic identifiability. To address this problem, we will use an analytic approach in which we consider specific variations of Problem 1, for example by choosing a specific pair of factors $(\boldsymbol{X}, \boldsymbol{Y})$ or a specific family of allowed pairs of supports $\hat{\Omega}$. In these variations, we assume that we have identifiability, and we derive some necessary conditions from this assumption. After obtaining a set of such necessary conditions, we will then try to find necessary and sufficient conditions for identifiability in the general case. In this section, we will only do the analysis for instance identifiability, since global identifiability already implies instance identifiability.

2.3.1 Identifiability when fixing one factor

In a first variation of Problem 1, we will consider the case where we fix one factor. Consider $\hat{\Omega} \subseteq \mathbb{B}^{n\times r} \times \mathbb{B}^{r\times m}$ a family of allowed pairs of supports. Let $(\boldsymbol{X}, \boldsymbol{Y}) \in \Sigma_{\hat{\Omega}}$ be a pair of factors of reference with an allowed support, and denote the observed matrix product $\boldsymbol{Z} := \boldsymbol{X}\boldsymbol{Y}$. We can then define the following linear inverse problem (2.24), which is the linear version of (2.23):

$$\begin{aligned} &\text{find} && \boldsymbol{B} \\ &\text{subject to} && \boldsymbol{X}\boldsymbol{B} = \boldsymbol{Z}, \\ & && (\boldsymbol{X}, \boldsymbol{B}) \in \Sigma_{\hat{\Omega}}. \end{aligned} \tag{2.24}$$

In (2.24), we only consider the case where we fix the left factor, and we can use matrix transpose to consider the case where the right factor is fixed. Then, we can look for conditions of uniqueness of the solution of (2.24), up to scaling and permutation equivalence, which leads to the notion of *right identifiability*.

Definition 2.5 (Right identifiability)**.** Let $\Omega_R \subseteq \mathbb{B}^{r\times m}$ be a family of allowed right supports, and consider a pair of factors $(\boldsymbol{X}, \boldsymbol{Y}) \in \mathbb{C}^{n\times r} \times \Sigma_{\Omega_R}$. We say that $\boldsymbol{Y}$ is right identifiable for $(\Omega_R, \boldsymbol{X})$ if, for all right factors $\boldsymbol{B} \in \Sigma_{\Omega_R}$ such that $\boldsymbol{X}\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{B}$, we have $(\boldsymbol{X}, \boldsymbol{B}) \sim (\boldsymbol{X}, \boldsymbol{Y})$.

For a family of allowed pairs of supports $\hat{\Omega} \subseteq \mathbb{B}^{n\times r} \times \mathbb{B}^{r\times m}$ and a fixed left factor $\boldsymbol{A} \in \mathbb{C}^{n\times r}$, we denote:

$$\Omega_R(\boldsymbol{A}) := \{\boldsymbol{S_R} \in \mathbb{B}^{r\times m} \mid (\mathrm{supp}(\boldsymbol{A}), \boldsymbol{S_R}) \in \hat{\Omega}\} \tag{2.25}$$

Then, Lemma 2.1 below states that right identifiability is a necessary condition of identifiability.

Lemma 2.1. *Let $\hat{\Omega} \subseteq \mathbb{B}^{n\times r} \times \mathbb{B}^{r\times m}$ be a family of allowed pairs of supports. Consider $(\boldsymbol{X_0}, \boldsymbol{Y_0}) \in \Sigma_{\hat{\Omega}}$ a pair of factors. Suppose that $(\boldsymbol{X}_0, \boldsymbol{Y}_0)$ is identifiable for the family $\hat{\Omega}$ (Definition 2.3). Then, for any pair of factors $(\boldsymbol{X}, \boldsymbol{Y}) \in [\boldsymbol{X_0}, \boldsymbol{Y_0}]$, the right factor $\boldsymbol{Y}$ is right identifiable for the family $\Omega_R(\boldsymbol{X})$ and the left factor $\boldsymbol{X}$ (Definition 2.5).*

Proof. Let $(\boldsymbol{X}, \boldsymbol{Y})$ be a pair equivalent to $(\boldsymbol{X_0}, \boldsymbol{Y_0})$, and $\boldsymbol{B} \in \Sigma_{\Omega_R(\boldsymbol{X})}$ such that $\boldsymbol{X}\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{B}$. Then, by definition of $\Omega_R(\boldsymbol{X})$ in (2.25), we have $(\mathrm{supp}(\boldsymbol{X}), \mathrm{supp}(\boldsymbol{B})) \in \hat{\Omega}$. In particular, $(\boldsymbol{X}, \boldsymbol{B}) \in \Sigma_{\hat{\Omega}}$ and $(\boldsymbol{X}, \boldsymbol{B})$ verifies $\boldsymbol{X}\boldsymbol{B} = \boldsymbol{X}\boldsymbol{Y} = \boldsymbol{X_0}\boldsymbol{Y_0}$. Then, by instance identifiability of $(\boldsymbol{X_0}, \boldsymbol{Y_0})$ for the family $\hat{\Omega}$, we have $(\boldsymbol{X}, \boldsymbol{B}) \sim (\boldsymbol{X_0}, \boldsymbol{Y_0})$, and $(\boldsymbol{X}, \boldsymbol{B}) \sim (\boldsymbol{X}, \boldsymbol{Y})$. □

In the case where the considered family of allowed pairs of supports $\hat{\Omega}$ is stable by permutation (Definition 2.2), Proposition 2.1 below extends the result of Lemma 2.1, because it states that in order to guarantee identifiability of a pair of factors (Definition 2.3), it suffices to show that the left factor can be identified up to equivalence (in the sense of condition 1 in the proposition) and that the right factor is right identifiable (Definition 2.5), after fixing the identified left factor. This proposition is actually the application of [16, Theorem 2.8] in the specific instance of matrix sparse factorization with two factors. Here, we show in the appendices a more direct proof of this theorem in the specific case of matrix sparse factorization with the group of scaled permutation matrices (Definition 2.1).

Proposition 2.1. ($\star$) *Let $\hat{\Omega} \subseteq \mathbb{B}^{n\times r} \times \mathbb{B}^{r\times m}$ be a family of allowed pairs of supports stable by permutation (Definition 2.2), and $(\boldsymbol{X}, \boldsymbol{Y}) \in \Sigma_{\hat{\Omega}}$ be a pair of factors. Then, $(\boldsymbol{X}, \boldsymbol{Y})$ is identifiable for the family $\hat{\Omega}$ (Definition 2.3) if, and only if, the following conditions are verified:*

1. *for all $(\boldsymbol{A}, \boldsymbol{B}) \in \Sigma_{\hat{\Omega}}$ verifying $\boldsymbol{A}\boldsymbol{B} = \boldsymbol{X}\boldsymbol{Y}$, there exists a scaled permutation matrix $\boldsymbol{C} \in \mathbb{C}^{r\times r}$ (Definition 2.1) such that $\boldsymbol{A} = \boldsymbol{X}\boldsymbol{C}$;*
2. *$\boldsymbol{Y}$ is right identifiable for the family $\Omega_R(\boldsymbol{X})$ and the left factor $\boldsymbol{X}$ (Definition 2.5).*

Proof sketch. Essentially, condition 1 means that the left factor can be identified, up to a scaled permutation matrix. The idea then is that we can fix this identified left factor, and condition 2 is precisely the identifiability of the right factor when the left factor is fixed. The formal proof is deferred to the appendices. □

Therefore, Lemma 2.1 and Proposition 2.1 justify why solving Problem 2 below can give necessary conditions of identifiability of a pair of factors.

Problem 2. Let $\Omega_R \subseteq \mathbb{B}^{r\times m}$ be a family of allowed right supports, and $(\boldsymbol{X}, \boldsymbol{Y}) \in \mathbb{C}^{n\times r} \times \Sigma_{\Omega_R}$ a pair of factors. Under which conditions on $(\boldsymbol{X}, \boldsymbol{Y})$ and Ω_R do we have right identifiability (Definition 2.5) of $\boldsymbol{Y}$ for the family Ω_R and the fixed left factor $\boldsymbol{X}$?

Non-degenerate linear inverse problem

In this paragraph, we focus on a more specific instance of Problem 2, where the fixed left factor $\boldsymbol{X}$ is not invariant to scaled permutations, in the sense of the following definition.

Definition 2.6 (Non-invariant left factor to scaled permutations)**.** Let $\Omega_R \subseteq \mathbb{B}^{r\times m}$ be a family of allowed right supports, and $\boldsymbol{X} \in \mathbb{C}^{n\times r}$ a left factor. We say that $\boldsymbol{X}$ is not invariant to scaled permutations for the family Ω_R, if for all right factors $\boldsymbol{B}, \boldsymbol{B'} \in \Sigma_{\Omega_R}$ such that $(\boldsymbol{X}, \boldsymbol{B}) \sim (\boldsymbol{X}, \boldsymbol{B'})$, we have $\boldsymbol{B} = \boldsymbol{B'}$.

Remark. Basically, when the left factor $\boldsymbol{X}$ is not invariant to scaled permutations for the family Ω_R, we can reduce the equivalence $(\boldsymbol{X}, \boldsymbol{B}) \sim (\boldsymbol{X}, \boldsymbol{B})$ to the equality $\boldsymbol{B} = \boldsymbol{B'}$. The characterization of this property is not studied in this report, and is left as a future work. In the following, we will usually say that such $\boldsymbol{X}$ is *non-degenerate*.

Then, given a family of allowed right supports Ω_R, in the specific case where the left factor $\boldsymbol{X}$ is non-degenerate, right identifiability for $\hat{\Omega}$ and $\boldsymbol{X}$ implies a more specific notion of identifiability, which will be referred to as *exact right identifiability*.

Definition 2.7 (Exact right identifiability)**.** Let $\Omega_R \subseteq \mathbb{B}^{r\times m}$ be a family of allowed right supports, and consider a pair of factors $(\boldsymbol{X}, \boldsymbol{Y}) \in \mathbb{C}^{n\times r} \times \Sigma_{\Omega_R}$. We say that $\boldsymbol{Y}$ is exactly right identifiable for the family $(\Omega_R, \boldsymbol{X})$ if, for all right factors $\boldsymbol{B} \in \Sigma_{\Omega_R}$ such that $\boldsymbol{XY} = \boldsymbol{XB}$, we have $\boldsymbol{B} = \boldsymbol{Y}$.

Remark. We use the term "exact" in this definition to highlight that we require the equality $\boldsymbol{B} = \boldsymbol{Y}$ instead of the equivalence $(\boldsymbol{X}, \boldsymbol{B}) \sim (\boldsymbol{X}, \boldsymbol{Y})$.

Lemma 2.2 shows the equivalence between *right identifiability* (Definition 2.5) and *exact right identifiability* (Definition 2.7) in the case where the left factor $\boldsymbol{X}$ is non-degenerate.

Lemma 2.2. *Let $\Omega_R \subseteq \mathbb{B}^{r\times m}$ be a family of allowed right supports, and $(\boldsymbol{X}, \boldsymbol{Y}) \in \mathbb{C}^{n\times r} \times \Sigma_{\Omega_R}$ a pair of factors. Suppose that $\boldsymbol{X}$ is not invariant to scaled permutations for the family Ω_R (Definition 2.6). Then, $\boldsymbol{Y}$ is right identifiable for $(\Omega_R, \boldsymbol{X})$ (Definition 2.5) if, and only if, $\boldsymbol{Y}$ is exactly right identifiable for $(\Omega_R, \boldsymbol{X})$ (Definition 2.7).*

Proof. Suppose that $\boldsymbol{Y}$ is right identifiable for $(\Omega_R, \boldsymbol{X})$. Let $\boldsymbol{B} \in \Sigma_{\Omega_R}$ such that $\boldsymbol{XB} = \boldsymbol{XY}$. Then, by right identifiability of $\boldsymbol{Y}$ for $(\Omega_R, \boldsymbol{X})$, we have $(\boldsymbol{X}, \boldsymbol{B}) \sim (\boldsymbol{X}, \boldsymbol{Y})$. But since $\boldsymbol{X}$ is not invariant to scaled permutations, we have $\boldsymbol{B} = \boldsymbol{Y}$, which means that $\boldsymbol{Y}$ is exactly right identifiable for $(\Omega_R, \boldsymbol{X})$. The converse is true since $\boldsymbol{B} = \boldsymbol{Y}$ implies $(\boldsymbol{X}, \boldsymbol{B}) \sim (\boldsymbol{X}, \boldsymbol{Y})$, for any right factors $\boldsymbol{B} \in \mathbb{C}^{r\times m}$. □

As we will see in Chapter 3, it is easier to characterize exact right identifiability (Definition 2.7) instead of right identifiability (Definition 2.5) in general. Therefore, in this work, we will give partial solutions to Problem 2 by considering only the case where the left factor $\boldsymbol{X}$ is not invariant to scaled permutations, and leave the case where the left factor $\boldsymbol{X}$ is invariant to scaled permutations as an open question.

2.3.2 Identifiability when fixing a pair of supports

Another interesting variation of Problem 1 is the one where we fix a pair of supports, in the sense that the family of allowed pairs of supports is reduced to one pair of supports. In this case, consider $\hat{\boldsymbol{S}} \in \mathbb{B}^{n\times r} \times \mathbb{B}^{r\times m}$ a fixed pair of supports, $(\boldsymbol{X}, \boldsymbol{Y}) \in \Sigma_{\hat{\boldsymbol{S}}}$ a pair of supports, and denote $\boldsymbol{Z} := \boldsymbol{XY}$

the observed product matrix. Then, we can formulate the following bilinear inverse problem (2.26), which is a specific instance of the general bilinear inverse problem (2.23):

$$\begin{aligned} \text{find} \quad & (\boldsymbol{A}, \boldsymbol{B}) \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{B} = \boldsymbol{Z}, \\ & (\boldsymbol{A}, \boldsymbol{B}) \in \Sigma_{\hat{\boldsymbol{S}}}. \end{aligned} \tag{2.26}$$

Therefore, we can define from (2.26) a notion of *fixed-support identifiability* (Definition 2.8).

Definition 2.8 (Fixed-support identifiability)**.** Let $(\boldsymbol{X}, \boldsymbol{Y}) \in \mathbb{C}^{n \times r} \times \mathbb{C}^{r \times m}$ be a pair of factors. We say that $(\boldsymbol{X}, \boldsymbol{Y})$ is identifiable with fixed support if $(\boldsymbol{X}, \boldsymbol{Y})$ is identifiable (Definition 2.3) for the family $\{(\operatorname{supp}(\boldsymbol{X}), \operatorname{supp}(\boldsymbol{Y}))\}$.

We remark that the family $\{\hat{\boldsymbol{S}}\}$ for a given pair of supports $\hat{\boldsymbol{S}}$ is not stable by permutation in general, in the sense of Definition 2.2. However, thanks to Lemma 2.3 below, a pair of factors $(\boldsymbol{X}, \boldsymbol{Y})$ is identifiable for the family $\{(\operatorname{supp}(\boldsymbol{X}), \operatorname{supp}(\boldsymbol{Y}))\}$ if, and only if, $(\boldsymbol{X}, \boldsymbol{Y})$ is identifiable for the family $[\operatorname{supp}(\boldsymbol{X}), \operatorname{supp}(\boldsymbol{Y})]$. In other words, considering fixed-support identifiability (Definition 2.8) is not in contradiction with the discussion in Section 2.1.3 mentioning that we only consider families of allowed pairs of supports stable by permutation.

Lemma 2.3. $(\star)$ *Let $\hat{\boldsymbol{S}} \in \mathbb{B}^{n \times r} \times \mathbb{B}^{r \times m}$ be a fixed pair of supports, and consider a pair of factors $(\boldsymbol{X}, \boldsymbol{Y}) \in \Sigma_{\hat{\boldsymbol{S}}}$. Then, $(\boldsymbol{X}, \boldsymbol{Y})$ is identifiable for the family $[\hat{\boldsymbol{S}}]$ if, and only if, $(\boldsymbol{X}, \boldsymbol{Y})$ is identifiable for the family $\{\hat{\boldsymbol{S}}\}$.*

Proof. The proof is deferred to the appendices. □

Therefore, we can formulate Problem 3 which is a specific instance of Problem 1.

Problem 3. Let $(\boldsymbol{X}, \boldsymbol{Y}) \in \mathbb{C}^{n \times r} \times \mathbb{C}^{r \times r}$ be a pair of factors. Under which conditions on $(\boldsymbol{X}, \boldsymbol{Y})$ do we have fixed-support identifiability (Definition 2.8) of $(\boldsymbol{X}, \boldsymbol{Y})$?

2.4 Summary

Based on the previous analysis, we summarize here the critical questions that we would like to solve in this work.

1. What is the characterization of exact right identifiability (Definition 2.7)?
2. What is the characterization of fixed-support identifiability (Definition 2.8)?

Figure 2.1 is a summary to help the reader to have a global view over the different definitions of identifiability mentioned in this report. These notions have been introduced for an analytic purpose, and are necessary conditions of identifiability in a general case where any family of allowed pairs of supports is considered. Some of the definitions have not been introduced yet, and they will be discussed in the next chapters. Chapter 3 will focus on the problem variation where the left factor is fixed (characterization of right identifiaiblity), while Chapter 4 will focus on the problem variation where the pair of supports is fixed (characterization of entry values identifiability). We highlight in Figure 2.1 notions of identifiability that have been characterized in this work.

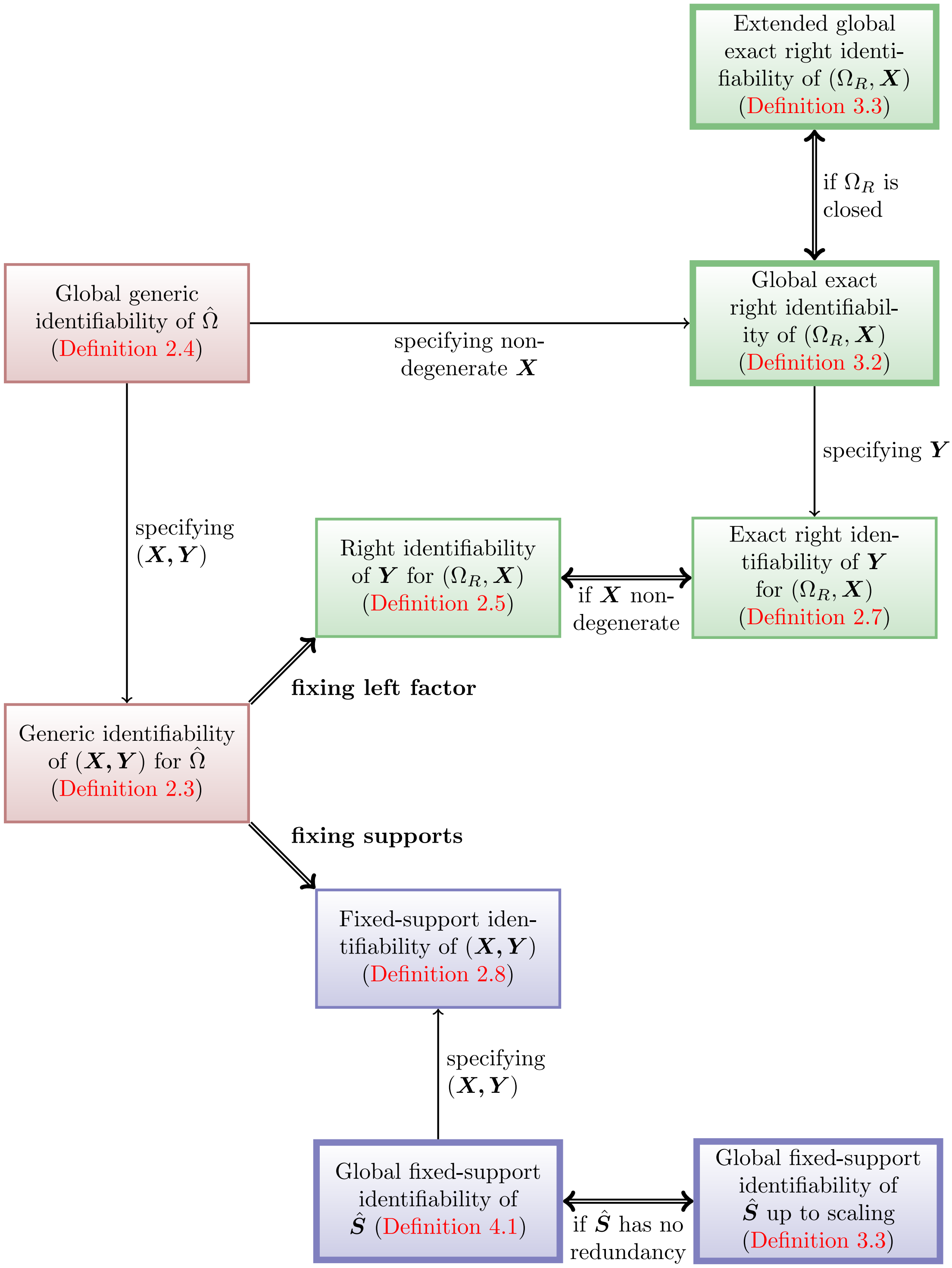


Figure 2.1: Relation between different notions of identifiability introduced in this work. Each box color corresponds to a specific problem: red for the generic problem (Problem 1), green for the problem variation where the left factor is fixed (Problem 2), and blue for the problem variation where the pair of supports is fixed (Problem 3). Thick boxes correspond to notions of identifiability that have been characterized in this work.

Chapter 3

Right identifiability results

The objective of this chapter is to address Problem 2. We give here a global version of the definition of right identifiability (Definition 2.5).

Definition 3.1 (Global right identifiability)**.** Let $\Omega_R \subseteq \mathbb{B}^{r\times m}$ be a family of allowed right supports, and $\boldsymbol{X} \in \mathbb{C}^{n\times r}$ a left factor. We say that $(\Omega_R, \boldsymbol{X})$ is globally right identifiable, if for any $\boldsymbol{Y}, \boldsymbol{Y'} \in \Sigma_{\Omega_R}$ verifying $\boldsymbol{XY} = \boldsymbol{XY'}$, we have $(\boldsymbol{X}, \boldsymbol{Y}) \sim (\boldsymbol{X}, \boldsymbol{Y'})$.

However, we will focus on the specific case where the left factor $\boldsymbol{X}$ is not permutation invariant to scaled permutations (Definition 2.6). The idea is to characterize *global exact right identifiability*, which is the global version of exact right identifiability (Definition 2.7).

Definition 3.2 (Global exact right identifiability)**.** Let $\Omega_R \subseteq \mathbb{B}^{r\times m}$ be a family of allowed right supports, and $\boldsymbol{X} \in \mathbb{C}^{n\times r}$ a left factor. We say that $(\Omega_R, \boldsymbol{X})$ is globally and exactly right identifiable, if for any $\boldsymbol{Y}, \boldsymbol{Y'} \in \Sigma_{\Omega_R}$ verifying $\boldsymbol{XY} = \boldsymbol{XY'}$, we have $\boldsymbol{Y} = \boldsymbol{Y'}$.

Remark. This is the global version of instance exact right identifiability defined in Definition 2.7 in the sense that $(\Omega_R, \boldsymbol{X})$ is globally and exactly right identifiable if, and only if, for all right factors $\boldsymbol{Y} \in \Sigma_{\Omega_R}$, $\boldsymbol{Y}$ is exactly right identifiable for $(\Omega_R, \boldsymbol{X})$.

In the same spirit of Lemma 2.1, we can show in Lemma 3.1 below that global exact right identifiability (Definition 3.2) is a necessary condition of global generic identifiability (Definition 2.4).

Lemma 3.1. *Let $\hat{\Omega} \subseteq \mathbb{B}^{n\times r} \times \mathbb{B}^{r\times m}$ be a family of allowed pairs of supports. Suppose that $\hat{\Omega}$ is globally and generically identifiable (Definition 2.4). Then, for any left factor $\boldsymbol{X} \in \mathbb{C}^{n\times r}$ which is not invariant to scaled permutations for the family $\Omega_R(\boldsymbol{X})$ (Definition 2.6), $(\Omega_R(\boldsymbol{X}), \boldsymbol{X})$ is globally and exactly right identifiable (Definition 3.2).*

Remark. Recall that the set $\Omega_R(\boldsymbol{A})$ is defined in (2.25), for any left factor $\boldsymbol{A} \in \mathbb{C}^{n\times r}$ and family $\hat{\Omega} \subseteq \mathbb{B}^{n\times r} \times \mathbb{B}^{r\times m}$.

Proof. Let $\boldsymbol{X} \in \mathbb{C}^{n\times r}$ be a left factor which is not invariant to scaled permutations for the family $\Omega_R(\boldsymbol{X})$, and $\boldsymbol{Y}, \boldsymbol{Y'} \in \Sigma_{\Omega_R(\boldsymbol{X})}$ be two right factors such that $\boldsymbol{XY} = \boldsymbol{XY'}$. By definition of $\Omega_R(\boldsymbol{X})$ given in (2.25), we have $(\boldsymbol{X}, \boldsymbol{Y}), (\boldsymbol{X}, \boldsymbol{Y'}) \in \Sigma_{\hat{\Omega}}$. Then, by assumption on $\hat{\Omega}$, we obtain $(\boldsymbol{X}, \boldsymbol{Y}) \sim (\boldsymbol{X}, \boldsymbol{Y'})$. But $\boldsymbol{X}$ is not invariant to scaled permutation for the family $\Omega_R(\boldsymbol{X})$, so $\boldsymbol{Y} = \boldsymbol{Y'}$. This shows that $(\Omega_R(\boldsymbol{X}), \boldsymbol{X})$ is globally and exactly right identifiable. □

3.1 Linearization of the inverse problem

When studying right identifiability, the left factor is fixed, so that the problem is reduced to a linear inverse problem. Therefore, we formulate the problem of right identifiability as a linear inverse problem, by vectorizing matrices. For any integers p, q, we define the *vectorization* function

$\text{vec}_{p,q} : \mathbb{C}^{p\times q} \to \mathbb{C}^{pq}$ where for $\boldsymbol{M} \in \mathbb{C}^{p\times q}$, $\text{vec}_{p,q}(\boldsymbol{M})$ is the vector for which the $((j-1)p+i)$-th element is $\boldsymbol{M}_{ij}$, for all $(i,j) \in [\![p]\!] \times [\![q]\!]$. In other words, we have:

$$\forall \boldsymbol{M} \in \mathbb{C}^{p\times q}, \quad \text{vec}_{p,q}(\boldsymbol{M}) = \begin{pmatrix} \boldsymbol{M}_{\bullet 1} \\ \boldsymbol{M}_{\bullet 2} \\ \vdots \\ \boldsymbol{M}_{\bullet q} \end{pmatrix} \in \mathbb{C}^{pq} \tag{3.1}$$

When there is no ambiguity, we omit the subscripts p, q and write vec instead of $\text{vec}_{p,q}$. This vectorization operator allows us to express the matrix product between two matrices as a matrix-vector multiplication. More precisely, we have the following lemma (Lemma 3.2).

Lemma 3.2. ($\star$) *The following assertions are verified:*

1. *for any* $(\boldsymbol{X}, \boldsymbol{Y}) \in \mathbb{C}^{n\times r} \times \mathbb{C}^{r\times m}$, *we have* $(\boldsymbol{I_m} \otimes \boldsymbol{X})\,\text{vec}(\boldsymbol{Y}) = \text{vec}(\boldsymbol{XY})$;
2. *for any* $\boldsymbol{X} \in \mathbb{C}^{n\times r}$, $\boldsymbol{y} \in \mathbb{C}^{rm}$, *we have* $\text{vec}^{-1}\left((\boldsymbol{I_m} \otimes \boldsymbol{X})\boldsymbol{y}\right) = \boldsymbol{X}\,\text{vec}^{-1}(\boldsymbol{y})$.

Proof. The first part is obtained directly by applying the definition of the matrix product, and the second part is a direct consequence of the first part. The formal proof is deferred to the appendices. □

We will use this vectorization to formulate conditions of global exact right identifiability (Definition 3.2). Throughout this chapter, we will rely on the following useful lemma (Lemma 3.3) which simplifies the expression of a secant set. In this lemma, there will be the following abuse of notation: for any vector supports $\boldsymbol{s}, \boldsymbol{s'} \in \mathbb{B}^p$, we denote $\boldsymbol{s} \cup \boldsymbol{s'} \in \mathbb{B}^p$ the vector support corresponding to the union of the corresponding sets, defined as:

$$\forall i \in [\![p]\!], \quad (\boldsymbol{s} \cup \boldsymbol{s'})_i := \begin{cases} 1 & \text{if } i \in \boldsymbol{s} \cup \boldsymbol{s'} \\ 0 & \text{otherwise} \end{cases}. \tag{3.2}$$

We use the same abuse of notation for matrices.

Lemma 3.3. ($\star$) *Let* $\boldsymbol{s}, \boldsymbol{s'} \in \mathbb{B}^p$ *be two vector supports. Then, the following assertions are verified:*

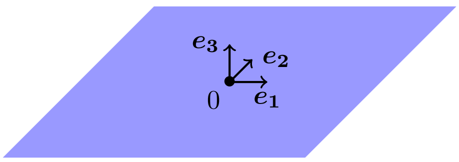
1. $\overline{\Sigma}_{\boldsymbol{s}} - \overline{\Sigma}_{\boldsymbol{s'}} = \overline{\Sigma}_{\boldsymbol{s}\cup\boldsymbol{s'}}$;
2. $\Sigma_{\boldsymbol{s}} - \Sigma_{\boldsymbol{s'}} = \overline{\Sigma}_{\boldsymbol{s}\cup\boldsymbol{s'}} \setminus \left(\bigcup_{i\in \boldsymbol{s}\Delta\boldsymbol{s'}} \text{span}(\boldsymbol{e_i})^{\perp}\right) := E_{\boldsymbol{s},\boldsymbol{s'}}$.

Proof. The proof is deferred to the appendices. □

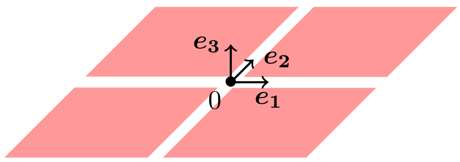
Remark. We observe that in assertion 2, the set $E_{\boldsymbol{s},\boldsymbol{s'}}$ is "almost" equal to $\overline{\Sigma}_{\boldsymbol{s}\cup\boldsymbol{s'}}$. Indeed, we have:

$$E_{\boldsymbol{s},\boldsymbol{s'}} = \{\boldsymbol{v} \in \mathbb{C}^p \mid \boldsymbol{v}_i = 0 \text{ for all } i \in [\![p]\!]\setminus(\boldsymbol{s}\cup\boldsymbol{s'}), \text{ and } \boldsymbol{v}_i \neq 0 \text{ for all } i \in \boldsymbol{s}\Delta\boldsymbol{s'}\} \tag{3.3}$$

We give an illustration in Figure 3.1.



(a) $\overline{\Sigma}_{\boldsymbol{s}\cup\boldsymbol{s'}}$



(b) $\Sigma_{\boldsymbol{s}} - \Sigma_{\boldsymbol{s'}} = E_{\boldsymbol{s},\boldsymbol{s'}}$

Figure 3.1: Representation in $\mathbb{R}^3$ of $\overline{\Sigma}_{\boldsymbol{s}\cup\boldsymbol{s'}}$ (in blue) and $\Sigma_{\boldsymbol{s}} - \Sigma_{\boldsymbol{s'}}$ (in red), where $\boldsymbol{s} = \boldsymbol{e_1}$, $\boldsymbol{s'} = \boldsymbol{e_2}$, and $(\boldsymbol{e_i})_{i=1}^3$ is the canonical basis of $\mathbb{R}^3$.

To explain how to characterize global exact right identifiability (Definition 3.2), we start in Section 3.2 with the specific case where the family of allowed right supports Ω_R is closed. After illustrating this characterization on concrete examples, we will generalize in Section 3.3 this characterization to the case where the family Ω_R is not necessarily closed.

3.2 Closed family of allowed right supports

We consider here the specific case where the family of allowed right supports Ω_R is closed, *i.e.*, we have $\overline{\Omega}_R = \Omega_R$ in the sense of (2.11). This leads to the introduction of a specific definition of global exact right identifiability, referred to as *extended global exact right identifiability*.

Definition 3.3 (Extended global exact right identifiability). Let $\Omega_R \subseteq \mathbb{B}^{r\times m}$ be a family of allowed right supports, and $\boldsymbol{X} \in \mathbb{C}^{n\times r}$ a left factor. We say that $(\Omega_R, \boldsymbol{X})$ is globally and exactly right identifiable with extension, if for any $\boldsymbol{Y}, \boldsymbol{Y'} \in \overline{\Sigma}_{\Omega_R}$ verifying $\boldsymbol{XY} = \boldsymbol{XY'}$, we have $\boldsymbol{Y} = \boldsymbol{Y'}$.

Remark. The only difference of this definition from Definition 3.2 is that we consider the extended model set $\overline{\Sigma}_{\Omega_R}$ instead of non-extended model set Σ_{Ω_R}.

Definition 3.2 and Definition 3.3 are actually equivalent in the specific case where the family of allowed right supports Ω_R is closed, as it is claimed by the following lemma.

Lemma 3.4. *Let $\Omega_R \subseteq \mathbb{B}^{r\times m}$ be a family of allowed right supports. Then, for any $\boldsymbol{X} \in \mathbb{C}^{n\times r}$, $(\Omega_R, \boldsymbol{X})$ is globally and exactly right identifiable with extension (Definition 3.3) if, and only if, $(\overline{\Omega}_R, \boldsymbol{X})$ is globally and exactly right identifiable (Definition 3.2).*

Proof. The equivalence is given by equation (2.12). □

3.2.1 Characterization of extended global exact right identifiability

The objective here is to characterize extended global exact right identifiability (Definition 3.3). Using the vectorization operator, introduced in (3.1), and Lemma 3.2, the matrix product $\boldsymbol{XY} = \boldsymbol{Z}$ is transformed into the matrix-vector multiplication $(\boldsymbol{I_m} \otimes \boldsymbol{X})\,\mathrm{vec}(\boldsymbol{Y}) = \mathrm{vec}(\boldsymbol{Z})$, and Proposition 3.1 reduces extended global exact right identifiability (Definition 3.3) to the injectivity of the matrix $(\boldsymbol{I_m} \otimes \boldsymbol{X})$, when restricting the linear operation to specific secant sets. This vectorization process allows us to have a similar framework as the one in [12, Theorem 2.13], but instead of considering the specific case of global sparsity (vectors are sparse if they have at most s nonzero entries, for a parameter s), we consider any sparsity model, depending on the choice of the family of allowed right supports. In other words, Proposition 3.1 and Theorem 3.1 below are the generalization of [12, Theorem 2.13] in the framework of model-based compressive sensing.

Proposition 3.1. ($\star$) *Let $\Omega_R \subseteq \mathbb{B}^{r\times m}$ be a family of allowed right supports, and $\boldsymbol{X} \in \mathbb{C}^{n\times r}$ a fixed left factor. Then the following assertions are equivalent:*

1. *$(\Omega_R, \boldsymbol{X})$ is globally and exactly right identifiable with extension (Definition 3.3);*
2. *for all $\boldsymbol{S_R}, \boldsymbol{S_R}' \in \Omega_R$, we have $\mathrm{Ker}(\boldsymbol{I_m} \otimes \boldsymbol{X}) \cap \left(\overline{\Sigma}_{\mathrm{vec}(\boldsymbol{S_R})} - \overline{\Sigma}_{\mathrm{vec}(\boldsymbol{S_R}')}\right) \subseteq \{0\}$.*

Proof. Suppose 1 and we want to prove 2. Let $\boldsymbol{S_R}, \boldsymbol{S_R}' \in \Omega_R$, and $\boldsymbol{y} \in \mathrm{Ker}(\boldsymbol{I_m} \otimes \boldsymbol{X}) \cap \left(\overline{\Sigma}_{\mathrm{vec}(\boldsymbol{S_R})} - \overline{\Sigma}_{\mathrm{vec}(\boldsymbol{S_R}')}\right)$. By definition, $(\boldsymbol{I_m} \otimes \boldsymbol{X})\boldsymbol{y} = 0$, and there exists $\boldsymbol{u} \in \overline{\Sigma}_{\mathrm{vec}(\boldsymbol{S_R})}$ and $\boldsymbol{v} \in \overline{\Sigma}_{\mathrm{vec}(\boldsymbol{S_R}')}$ such that $\boldsymbol{y} = \boldsymbol{u} - \boldsymbol{v}$. This means that $(\boldsymbol{I_m} \otimes \boldsymbol{X})\boldsymbol{u} = (\boldsymbol{I_m} \otimes \boldsymbol{X})\boldsymbol{v}$. But by Lemma 3.2, this means that $\boldsymbol{X}\,\mathrm{vec}^{-1}(\boldsymbol{u}) = \boldsymbol{X}\,\mathrm{vec}^{-1}(\boldsymbol{v})$. Since $\mathrm{vec}^{-1}(\boldsymbol{u}) \in \overline{\Sigma}_{\boldsymbol{S_R}}$ and $\mathrm{vec}^{-1}(\boldsymbol{v}) \in \overline{\Sigma}_{\boldsymbol{S_R}'}$, we conclude by assumption 1 that $\mathrm{vec}^{-1}(\boldsymbol{u}) = \mathrm{vec}^{-1}(\boldsymbol{v})$, which leads to $\boldsymbol{u} = \boldsymbol{v}$ and $\boldsymbol{y} = 0$.

Conversely, suppose 2 and we want to prove 1. Let $\boldsymbol{Y}, \boldsymbol{Y'} \in \overline{\Sigma}_{\Omega_R}$ verifying $\boldsymbol{XY} = \boldsymbol{XY'}$. This means, by Lemma 3.2, that $(\boldsymbol{I_m} \otimes \boldsymbol{X})\,\mathrm{vec}(\boldsymbol{Y}) = (\boldsymbol{I_m} \otimes \boldsymbol{X})\,\mathrm{vec}(\boldsymbol{Y'})$, and $\mathrm{vec}(\boldsymbol{Y}) - \mathrm{vec}(\boldsymbol{Y'}) \in \mathrm{Ker}(\boldsymbol{I_m} \otimes \boldsymbol{X})$. Moreover, we have $\mathrm{vec}(\boldsymbol{Y}) \in \overline{\Sigma}_{\mathrm{vec}(\mathrm{supp}(\boldsymbol{Y}))}$ and $\mathrm{vec}(\boldsymbol{Y'}) \in \overline{\Sigma}_{\mathrm{vec}(\mathrm{supp}(\boldsymbol{Y'}))}$, which means that $\mathrm{vec}(\boldsymbol{Y}) - \mathrm{vec}(\boldsymbol{Y'}) \in \overline{\Sigma}_{\mathrm{vec}(\mathrm{supp}(\boldsymbol{Y}))} - \overline{\Sigma}_{\mathrm{vec}(\mathrm{supp}(\boldsymbol{Y'}))}$. This leads to $\mathrm{vec}(\boldsymbol{Y}) - \mathrm{vec}(\boldsymbol{Y'}) \in \mathrm{Ker}(\boldsymbol{I_m} \otimes \boldsymbol{X}) \cap \left(\overline{\Sigma}_{\mathrm{vec}(\mathrm{supp}(\boldsymbol{Y}))} - \overline{\Sigma}_{\mathrm{vec}(\mathrm{supp}(\boldsymbol{Y'}))}\right)$. By assumption 2, we conclude that $\mathrm{vec}(\boldsymbol{Y}) - \mathrm{vec}(\boldsymbol{Y'}) = 0$, and $\boldsymbol{Y} = \boldsymbol{Y'}$. □

Remark. There exists an alternative proof of this proposition, by considering it as a corollary of the more general Proposition A.1 presented in the appendices. The proof of this alternative proof is given in the appendices.

We remark that for any $\boldsymbol{S_R}, \boldsymbol{S'_R} \in \mathbb{B}^{r\times m}$, we have:

$$\begin{aligned}\overline{\Sigma}_{\mathrm{vec}(\boldsymbol{S_R})} - \overline{\Sigma}_{\mathrm{vec}(\boldsymbol{S_R}')} &= \overline{\Sigma}_{\mathrm{vec}(\boldsymbol{S_R})\cup\mathrm{vec}(\boldsymbol{S_R}')} \text{ by the first assertion of Lemma 3.3} \\ &= \overline{\Sigma}_{\mathrm{vec}(\boldsymbol{S_R}\cup\boldsymbol{S_R}')} \text{ since } \mathrm{vec}(\boldsymbol{S_R})\cup\mathrm{vec}(\boldsymbol{S_R}') = \mathrm{vec}(\boldsymbol{S_R}\cup\boldsymbol{S_R}'),\end{aligned} \tag{3.4}$$

so it is possible to simplify condition 2 of Proposition 3.1, by considering $\boldsymbol{S_R}\cup\boldsymbol{S_R}'$ as an element of $\mathbb{B}^{r\times m}$. Indeed, because of the block structure of the matrix $(\boldsymbol{I_m}\otimes\boldsymbol{X})$, we have the following lemma.

Lemma 3.5. ($\star$) *Let $\boldsymbol{S_R}\in\mathbb{B}^{r\times m}$ and $\boldsymbol{X}\in\mathbb{C}^{n\times r}$. The following assertions are equivalent:*

1. $\mathrm{Ker}(\boldsymbol{I_m}\otimes\boldsymbol{X})\cap\overline{\Sigma}_{\mathrm{vec}(\boldsymbol{S_R})} = \{0\}$;
2. *for all $l\in[\![m]\!]$, we have* $\mathrm{Ker}(\boldsymbol{X})\cap\overline{\Sigma}_{(\boldsymbol{S_R})_{\bullet l}} = \{0\}$;
3. *for all $l\in[\![m]\!]$, we have* $\mathrm{Ker}\left(\boldsymbol{X}_{|[\![n]\!]\times(\boldsymbol{S_R})_{\bullet l}}\right) = \{0\}$;
4. *for all $l\in[\![m]\!]$, the columns $\{\boldsymbol{X}_{\bullet l} \mid l\in(\boldsymbol{S_R})_{\bullet l}\}$ are linearly independent.*

Proof sketch. Essentially, given a matrix $\boldsymbol{M}\in\mathbb{C}^{p\times q}$ and a subset of indices $T\subseteq[\![q]\!]$, the condition $\mathrm{Ker}(\boldsymbol{M})\cap\overline{\Sigma}_T = \{0\}$ is essentially the linear independence of columns in $\boldsymbol{M}$ indexed by T. Here, the lemma uses the block structure of $(\boldsymbol{I_m}\otimes\boldsymbol{X})$ to reformulate the linear independence of columns in $(\boldsymbol{I_m}\otimes\boldsymbol{X})$ indexed by $\mathrm{vec}(\boldsymbol{S_R})$ with the linear independence of several subsets of columns in $\boldsymbol{X}$. The formal proof is deferred to the appendices. ☐

Then we can characterize global exact right identifiability in the specific case where the family of right supports is closed (which is equivalent to the notion of global exact right identifiability with extension given by Definition 3.3), using simply linear independence of specific columns of $\boldsymbol{X}$.

Theorem 3.1. *Let $\Omega_R\subseteq\mathbb{B}^{r\times m}$ be a closed family of allowed right supports, in the sense that $\Omega_R = \overline{\Omega}_R$, and $\boldsymbol{X}\in\mathbb{C}^{n\times r}$ a fixed left factor. Denote:*

$$\mathcal{T} := \{(\boldsymbol{S_R})_{\bullet l}\cup(\boldsymbol{S_R}')_{\bullet l} \mid \boldsymbol{S_R},\boldsymbol{S'_R}\in\Omega_R,\ l\in[\![m]\!]\}. \tag{3.5}$$

Then the following assertions are equivalent:

1. $(\Omega_R,\boldsymbol{X})$ *is globally and exactly right identifiable (Definition 3.2);*
2. $(\Omega_R,\boldsymbol{X})$ *is globally and exactly right identifiable with extension (Definition 3.3);*
3. *for all $\boldsymbol{S_R},\boldsymbol{S_R}'\in\Omega_R$, for all column indices $l\in[\![m]\!]$, we have* $\mathrm{Ker}\,\boldsymbol{X}\cap\overline{\Sigma}_{(\boldsymbol{S_R})_{\bullet l}\cup(\boldsymbol{S_R}')_{\bullet l}} = \{0\}$;
4. $\mathrm{Ker}(\boldsymbol{X})\cap\bigcup_{T\in\mathcal{T}}\overline{\Sigma}_T = \{0\}$;
5. *for all subsets $T\in\mathcal{T}$, the columns $\{\boldsymbol{X}_{\bullet l}\mid l\in T\}$ are linearly independent.*

Proof. The equivalence $1\Leftrightarrow 2$ is given by Lemma 3.4. Let $\boldsymbol{S_R},\boldsymbol{S'_R}\in\Omega_R$. Then, we have:

$$\begin{aligned} & \mathrm{Ker}(\boldsymbol{I_m}\otimes\boldsymbol{X})\cap\left(\overline{\Sigma}_{\mathrm{vec}(\boldsymbol{S_R})} - \overline{\Sigma}_{\mathrm{vec}(\boldsymbol{S_R}')}\right)\subseteq\{0\} \\ \iff & \mathrm{Ker}(\boldsymbol{I_m}\otimes\boldsymbol{X})\cap\overline{\Sigma}_{\mathrm{vec}(\boldsymbol{S_R})\cup\mathrm{vec}(\boldsymbol{S_R}')}\subseteq\{0\} \text{ by Lemma 3.3} \\ \iff & \mathrm{Ker}(\boldsymbol{I_m}\otimes\boldsymbol{X})\cap\overline{\Sigma}_{\mathrm{vec}(\boldsymbol{S_R})\cup\mathrm{vec}(\boldsymbol{S_R}')} = \{0\} \text{ because } \overline{\Sigma}_{\mathrm{vec}(\boldsymbol{S_R})\cup\mathrm{vec}(\boldsymbol{S_R}')} \text{ is a linear space} \\ \iff & \mathrm{Ker}(\boldsymbol{I_m}\otimes\boldsymbol{X})\cap\overline{\Sigma}_{\mathrm{vec}(\boldsymbol{S_R}\cup\boldsymbol{S_R}')} = \{0\} \text{ because } \mathrm{vec}(\boldsymbol{S_R})\cup\mathrm{vec}(\boldsymbol{S_R}') = \mathrm{vec}(\boldsymbol{S_R}\cup\boldsymbol{S_R}') \\ \iff & \forall l\in[\![m]\!],\ \mathrm{Ker}\,\boldsymbol{X}\cap\overline{\Sigma}_{(\boldsymbol{S_R}\cup\boldsymbol{S_R}')_{\bullet l}} = \{0\} \text{ by Lemma 3.5} \\ \iff & \forall l\in[\![m]\!],\ \mathrm{Ker}\,\boldsymbol{X}\cap\overline{\Sigma}_{(\boldsymbol{S_R})_{\bullet l}\cup(\boldsymbol{S_R}')_{\bullet l}} = \{0\} \text{ because } (\boldsymbol{S_R}\cup\boldsymbol{S_R}')_{\bullet l} = (\boldsymbol{S_R})_{\bullet l}\cup(\boldsymbol{S_R}')_{\bullet l},\end{aligned} \tag{3.6}$$

which shows the equivalence $2\Leftrightarrow 3$ by applying Proposition 3.1. We now show the equivalence $3\Leftrightarrow 4$. Suppose 3, and let $\boldsymbol{y}\in\mathrm{Ker}(\boldsymbol{X})\cap\bigcup_{T\in\mathcal{T}}\overline{\Sigma}_T$. Then, there exists $\boldsymbol{S_R},\boldsymbol{S'_R}\in\Omega_R$, and $l\in[\![m]\!]$ such that $\boldsymbol{y}\in\overline{\Sigma}_{(\boldsymbol{S_R})_{\bullet l}\cup(\boldsymbol{S_R}')_{\bullet l}}$. By applying assumption 3, we obtain $\boldsymbol{y}=0$. Conversely, suppose 4, and let $\boldsymbol{S_R},\boldsymbol{S'_R}\in\Omega_R$, $l\in[\![m]\!]$. Since we have the inclusion $\overline{\Sigma}_{(\boldsymbol{S_R})_{\bullet l}\cup(\boldsymbol{S_R}')_{\bullet l}}\subseteq\bigcup_{T\in\mathcal{T}}\overline{\Sigma}_T$, we conclude with assumption 4 that $\mathrm{Ker}(\boldsymbol{X})\cap\overline{\Sigma}_{(\boldsymbol{S_R})_{\bullet l}\cup(\boldsymbol{S_R}')_{\bullet l}} = \{0\}$. Finally, condition 5 is just a reformulation of condition 3, so $3\Leftrightarrow 5$. ☐

3.2.2 Examples of application

Condition 5 of Theorem 3.1 is easy to verify for some examples of application. In the following, we apply Theorem 3.1 to some examples of family of allowed right supports introduced in Example 2.1:

- right supports which are k-sparse by column, for some parameter k;
- right supports which are l-sparse by row, for some parameter l;
- right supports which are k-sparse by columns and l-sparse by row, for some parameters k, l;
- right supports which are globally s-sparse, for some parameter s.

Corollary 3.1. ($\star$) *Let $k \in [\![r]\!]$, and denote $\Omega_R := \{\boldsymbol{S} \in \mathbb{B}^{r\times m} \mid \|\boldsymbol{S}_{\bullet j}\|_0 \le k,\ \forall j \in [\![m]\!]\}$ the family of right supports which are k-sparse by column. Then, for any $\boldsymbol{X} \in \mathbb{C}^{n\times r}$, $(\Omega_R, \boldsymbol{X})$ is globally and exactly right identifiable (Definition 3.2) if, and only if, every subset of $\min(2k, r)$ columns of $\boldsymbol{X}$ is linearly independent.*

Corollary 3.2. ($\star$) *Let $l \in [\![m]\!]$, and denote $\Omega_R := \{\boldsymbol{S} \in \mathbb{B}^{r\times m} \mid \|\boldsymbol{S}_{i\bullet}\|_0 \le l,\ \forall i \in [\![p]\!]\}$ the family of right supports which are k-sparse by row. Then, for any $\boldsymbol{X} \in \mathbb{C}^{n\times r}$, $(\Omega_R, \boldsymbol{X})$ is globally and exactly right identifiable (Definition 3.2) if, and only if, all the columns of $\boldsymbol{X}$ are linearly independent.*

Corollary 3.3. ($\star$) *Let $(k, l) \in [\![r]\!] \times [\![m]\!]$, and denote $\Omega_R := \{\boldsymbol{S} \in \mathbb{B}^{r\times m} \mid \|\boldsymbol{S}_{\bullet j}\|_0 \le k,\ \|\boldsymbol{S}_{i\bullet}\|_0 \le l,\ \forall (i,j) \in [\![r]\!] \times [\![m]\!]\}$ the family of right supports which are k-sparse by column and l-sparse by row. Then, for any $\boldsymbol{X} \in \mathbb{C}^{n\times r}$, $(\Omega_R, \boldsymbol{X})$ is globally and exactly right identifiable (Definition 3.2) if, and only if, every subset of $\min(2k, r)$ columns of $\boldsymbol{X}$ is linearly independent.*

Corollary 3.4. ($\star$) *Let $s \in [\![rm]\!]$, and denote $\Omega_R := \{\boldsymbol{S} \in \mathbb{B}^{r\times m} \mid \|\boldsymbol{S}\|_0 \le s\}$ the family of right supports which are globally s-sparse. Then, for any $\boldsymbol{X} \in \mathbb{C}^{n\times r}$, $(\Omega_R, \boldsymbol{X})$ is globally and exactly right identifiable (Definition 3.2) if, and only if, every subset of $\min(2s, r)$ columns of $\boldsymbol{X}$ is linearly independent.*

Remark. It is necessary to have $n \ge r$ to have global exact right identifiability (Definition 3.2) of $(\Omega_R, \boldsymbol{X})$ when $2k \ge r$ for Corollary 3.1 and Corollary 3.3, when $2s \ge r$ for Corollary 3.4, and when l has any value for Corollary 3.2. Indeed, if $n < r$, it is not possible to have $\mathrm{rank}(\boldsymbol{X}) = r$, since $\mathrm{rank}(\boldsymbol{X}) \le \min(n, r)$.

Proof. The proofs are deferred to the appendices. They are the direct application of Theorem 3.1, by finding the explicit expression of the set $\mathcal{T}$ given in (3.5) in these specific choices of the family of allowed right supports, so that we can simplify condition 5 of the theorem in these cases. □

As we can see with these corollaries, we can use Theorem 3.1 for the majority of examples of family of allowed right supports, because they are usually closed.

3.3 General family of allowed right supports

However, for more generality, we might want to consider the case where the family of allowed right supports is not closed, *i.e.*, $\overline{\Omega}_R \neq \Omega_R$. The characterization of global exact right identifiability (Definition 3.2) in the general case is similar to the characterization of extended global exact right identifiability (Definition 3.3) presented in Section 3.2, but instead of considering extended model sets $\overline{\Sigma}_\bullet$, we consider non-extended model sets $\Sigma_\bullet$, which adds some complexity in the characterization. Despite of this complexity, one main reason to study the general case is to better understand some unavoidable mechanism behind identifiability when considering non-extended model set $\Sigma_\bullet$ instead of extended model set $\overline{\Sigma}_\bullet$. We will see that similar mechanism appears in the other problem variation considered in Chapter 4 where the pair of supports is fixed. This characterization in the general case is done in the appendices in Section A.1.

Chapter 4

Fixed-support identifiability results

This section addresses Problem 3. It is the specific instance of Problem 1 where the family of allowed pairs of supports is reduced to a singleton. Instead of the instance definition of fixed-support identifiability (Definition 2.8), we focus on the notion of *global fixed-support identifiability* defined below. This means that we are looking for conditions of fixed-support identifiability which depend only on the support, and not on a specific instance of pair of factors.

Definition 4.1 (Global fixed-support identifiability)**.** Let $\hat{\boldsymbol{S}} \in \mathbb{B}^{n\times r} \times \mathbb{B}^{r\times m}$ be a pair of supports. We say that $\hat{\boldsymbol{S}}$ is globally identifiable with fixed support if all pairs of factors $(\boldsymbol{X}, \boldsymbol{Y}) \in \Sigma_{\hat{\boldsymbol{S}}}$ are identifiable with fixed support (Definition 2.8); or, equivalently, if, for any pairs of factors $(\boldsymbol{X}, \boldsymbol{Y}), (\boldsymbol{X'}, \boldsymbol{Y'}) \in \Sigma_{\hat{\boldsymbol{S}}}$ verifying $\boldsymbol{XY} = \boldsymbol{X'Y'}$, we have $(\boldsymbol{X}, \boldsymbol{Y}) \sim (\boldsymbol{X'}, \boldsymbol{Y'})$.

4.1 Redundant structure in a pair of supports

In order to characterize global fixed-support identifiability (Definition 4.1), we distinguish two cases on the fixed pair of supports $\hat{\boldsymbol{S}} \in \mathbb{B}^{n\times r} \times \mathbb{B}^{r\times m}$, depending whether or not it has a *redundant structure*, in the sense of the following definition.

Definition 4.2 (Redundancy in a pair of supports)**.** Let $\hat{\boldsymbol{S}} \in \mathbb{B}^{n\times r} \times \mathbb{B}^{r\times m}$ be a pair of supports. We say that $\hat{\boldsymbol{S}}$ has a redundant structure if there exists two different indices $j_1, j_2 \in [\![r]\!]$ such that:

$$((\boldsymbol{S_L})_{\bullet j_1}, (\boldsymbol{S_R})_{j_1 \bullet}) = ((\boldsymbol{S_L})_{\bullet j_2}, (\boldsymbol{S_R})_{j_2 \bullet})\,. \tag{4.1}$$

Remark. In other words, a pair of supports has a redundancy structure if there exists two identical columns in the left support for two indices (j_1, j_2), and two identical rows in the right support for the same indices (j_1, j_2).

This property can also be characterized by using permutation of redundant columns in the left support and redundant rows in the right support, as it is shown in the following lemma.

Lemma 4.1. ($\star$) *Let $\hat{\boldsymbol{S}} \in \mathbb{B}^{n\times r} \times \mathbb{B}^{r\times m}$ be a pair of supports. Then, $\hat{\boldsymbol{S}}$ has a redundant structure if, and only if, there exists a permutation matrix $\boldsymbol{P} \in \mathbb{B}^{r\times r} \backslash \{\boldsymbol{I_r}\}$ different from the identity matrix such that $(\boldsymbol{S_L P}, \boldsymbol{P}^T \boldsymbol{S_R}) = (\boldsymbol{S_L}, \boldsymbol{S_R})$.*

Proof sketch. When a pair of supports presents a redundancy, permuting simultaneously the redundant columns in the left support and the redundant rows in the right support gives the same pair of supports. The formal proof is deferred to the appendices. □

Using this characterization, we show that, assuming a non-redundant structure in a pair of supports $\hat{\boldsymbol{S}}$, global fixed-support identifiability (Definition 4.1) of $\hat{\boldsymbol{S}}$ is equivalent to a specific notion of identifiability, referred to as *global fixed-support identifiability up to scaling*.

Definition 4.3 (Global fixed-support identifiability up to scaling)**.** Let $\hat{\boldsymbol{S}} \in \mathbb{B}^{n\times r} \times \mathbb{B}^{r\times m}$ be a pair of supports. We say that $\hat{\boldsymbol{S}}$ is globally identifiable with fixed support up to scaling if for any pairs of factors $(\boldsymbol{X},\boldsymbol{Y}),(\boldsymbol{X'},\boldsymbol{Y'}) \in \Sigma_{\hat{\boldsymbol{S}}}$ verifying $\boldsymbol{XY} = \boldsymbol{X'Y'}$, we have $(\boldsymbol{X},\boldsymbol{Y}) \sim_s (\boldsymbol{X'},\boldsymbol{Y'})$.

Remark. In this definition we use the term "up to scaling" to highlight the fact that we require the scaling equivalence $(\boldsymbol{X},\boldsymbol{Y}) \sim_s (\boldsymbol{A},\boldsymbol{B})$ instead of the general equivalence $(\boldsymbol{X},\boldsymbol{Y}) \sim (\boldsymbol{A},\boldsymbol{B})$, which includes scaling and permutation equivalence.

Lemma 4.2. *Let $\hat{\boldsymbol{S}} \in \mathbb{B}^{n\times r}\times\mathbb{B}^{r\times m}$ be a pair of supports. Suppose that $\hat{\boldsymbol{S}}$ does not have a redundant structure. Then, $\hat{\boldsymbol{S}}$ is globally identifiable with fixed support (Definition 4.1) if, and only if, $\hat{\boldsymbol{S}}$ is globally identifiable with fixed support up to scaling (Definition 4.3).*

Proof. Suppose that $\hat{\boldsymbol{S}}$ is globally identifiable with fixed support. Let $(\boldsymbol{X},\boldsymbol{Y}),(\boldsymbol{A},\boldsymbol{B}) \in \Sigma_{\hat{\boldsymbol{S}}}$ such that $\boldsymbol{AB} = \boldsymbol{XY}$. Then, by assumption, $(\boldsymbol{A},\boldsymbol{B})$ and $(\boldsymbol{X},\boldsymbol{Y})$ are equivalent in the sense of scaling and permutation equivalence. We fix $\boldsymbol{D}$ a diagonal matrix with nonzero entries on the diagonal and $\boldsymbol{P}$ a permutation matrix such that $\boldsymbol{ADP} = \boldsymbol{X}$ and $\boldsymbol{P}^T\boldsymbol{D}^{-1}\boldsymbol{B} = \boldsymbol{Y}$. Then, we have:

$$\begin{aligned}
(\boldsymbol{S_L},\boldsymbol{S_R}) &= \big(\operatorname{supp}(\boldsymbol{X}),\operatorname{supp}(\boldsymbol{Y})\big)\\
&= \big(\operatorname{supp}(\boldsymbol{ADP}),\operatorname{supp}(\boldsymbol{P}^T\boldsymbol{D}^{-1}\boldsymbol{B})\big)\\
&= \big(\operatorname{supp}(\boldsymbol{AD})\boldsymbol{P},\boldsymbol{P}^T\operatorname{supp}(\boldsymbol{D}^{-1}\boldsymbol{B})\big) \text{ since } \boldsymbol{P} \text{ is a permutation matrix}\\
&= \big(\operatorname{supp}(\boldsymbol{A})\boldsymbol{P},\boldsymbol{P}^T\operatorname{supp}(\boldsymbol{B})\big) \text{ since } \boldsymbol{D} \text{ has nonzero diagonal values}\\
&= \big(\boldsymbol{S_L}\boldsymbol{P},\boldsymbol{P}^T\boldsymbol{S_R}\big).
\end{aligned} \tag{4.2}$$

But $\hat{\boldsymbol{S}}$ does not have a redundant structure, so by contraposition of Lemma 4.1, $\boldsymbol{P}$ is necessary the identity matrix. This means that $(\boldsymbol{A},\boldsymbol{B}) \sim_s (\boldsymbol{X},\boldsymbol{Y})$, which shows that $\hat{\boldsymbol{S}}$ is globally identifiable with fixed support up to scaling. The converse is true because equivalence up to scaling $(\boldsymbol{A},\boldsymbol{B}) \sim_s (\boldsymbol{X},\boldsymbol{Y})$ implies equivalence up to scaling and permutation $(\boldsymbol{A},\boldsymbol{B}) \sim (\boldsymbol{X},\boldsymbol{Y})$. □

Therefore, Problem 3 in the case where the fixed pair of supports $\hat{\boldsymbol{S}}$ does not have redundancy is reduced to the characterization of global fixed-support identifiability up to scaling (Definition 4.3). The case where the fixed pair of supports $\hat{\boldsymbol{S}}$ has a redundant structure is covered by Section 4.3, while the opposite case is covered by Section 4.4.

4.2 Rank 1 contributions representation

To characterize global fixed-support identifiability (Definition 4.1), we will go back and forth between two representations for a pair of supports and for a pair of factors. Let $\hat{\boldsymbol{S}} \in \mathbb{B}^{n\times r} \times \mathbb{B}^{r\times m}$ be a pair of supports, and $(\boldsymbol{X},\boldsymbol{Y}) \in \mathbb{C}^{n\times r} \times \mathbb{C}^{r\times m}$ be a pair of factors. The two representations are:

1. the pair representation: $(\boldsymbol{S_L},\boldsymbol{S_R})$ for the pair of supports, and $(\boldsymbol{X},\boldsymbol{Y})$ for the pair of factors;
2. the rank 1 contributions representation [15]: $((\boldsymbol{S_L})_{\bullet i}(\boldsymbol{S_R})_{i\bullet})_{i=1}^r \in (\mathbb{B}^{n\times m} \cap \mathcal{R}_1)^r$ for the pair of supports, and $(\boldsymbol{X}_{\bullet i}\boldsymbol{Y}_{i\bullet})_{i=1}^r \in (\mathbb{C}^{n\times m} \cap \mathcal{R}_1)^r$ for the pair of factors.

Therefore, in the following, we define the rank 1 representation $(\boldsymbol{S_i})_{i=1}^r$ of the pair of supports $\hat{\boldsymbol{S}}$ as:

$$\forall i \in [\![r]\!], \quad \boldsymbol{S_i} := (\boldsymbol{S_L})_{\bullet i}(\boldsymbol{S_R})_{i\bullet} \in \mathbb{B}^{n\times m} \cap \mathcal{R}_1. \tag{4.3}$$

Then, for $\hat{\boldsymbol{S}} \in \mathbb{B}^{n\times r} \times \mathbb{B}^{r\times m}$, we will denote:

$$\underline{\boldsymbol{S}} := (\boldsymbol{S_i})_{i=1}^r \in \left(\mathbb{B}^{n\times m} \cap \mathcal{R}_1\right)^r. \tag{4.4}$$

For $i \in [\![r]\!]$, we will call $\boldsymbol{S_i}$ the i-th rank 1 support of the pair of supports $\hat{\boldsymbol{S}}$, and $(\boldsymbol{X}_{\bullet i}\boldsymbol{Y}_{i\bullet})_{i=1}^r$ the i-th rank 1 contribution of the pair of factors $(\boldsymbol{X}, \boldsymbol{Y})$. The rank 1 contributions representation is useful because it has the following important property:

$$\sum_{i=1}^{r} \boldsymbol{X}_{\bullet i}\boldsymbol{Y}_{i\bullet} = \boldsymbol{X}\boldsymbol{Y}. \tag{4.5}$$

$$\left(\begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{2} \\ \mathbf{4} & 0 & \mathbf{6} \\ 0 & \mathbf{2} & \mathbf{3} \\ 0 & 0 & \mathbf{1} \end{pmatrix}, \begin{pmatrix} \mathbf{4} & 0 & 0 & \mathbf{1} \\ \mathbf{1} & \mathbf{3} & \mathbf{2} & 0 \\ 0 & \mathbf{5} & \mathbf{1} & \mathbf{2} \end{pmatrix} \right)$$

(a) Pair representation $(\boldsymbol{X}, \boldsymbol{Y})$

$$\left(\begin{pmatrix} \mathbf{4} & 0 & 0 & \mathbf{1} \\ \mathbf{16} & 0 & 0 & \mathbf{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathbf{1} & \mathbf{3} & \mathbf{2} & 0 \\ 0 & 0 & 0 & 0 \\ \mathbf{2} & \mathbf{6} & \mathbf{4} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \mathbf{10} & \mathbf{2} & \mathbf{4} \\ 0 & \mathbf{30} & \mathbf{6} & \mathbf{12} \\ 0 & \mathbf{15} & \mathbf{3} & \mathbf{6} \\ 0 & \mathbf{5} & \mathbf{1} & \mathbf{2} \end{pmatrix} \right)$$

(b) Rank 1 contributions representation $(\boldsymbol{X}_{\bullet i}\boldsymbol{Y}_{i\bullet})_{i=1}^3$

Figure 4.1: Example of rank 1 contribution representations for a pair of factors $(\boldsymbol{X}, \boldsymbol{Y}) \in \mathbb{C}^{4\times 3} \times \mathbb{C}^{3\times 4}$. Nonzero entries are represented in bold.

Figure 4.1 illustrates the rank 1 contributions representation. Remark that the rank 1 contributions representation removes the scaling ambiguity, in the sense of Lemma 4.3 below, which claims that two pairs of factors are equivalent up to scaling equivalence if, and only if, they have the same rank 1 contribution representations.

Lemma 4.3. ($\star$) *Let* $(\boldsymbol{X}, \boldsymbol{Y}), (\boldsymbol{X'}, \boldsymbol{Y'}) \in \mathbb{C}^{n\times r} \times \mathbb{C}^{r\times m}$. *Then,* $(\boldsymbol{X}, \boldsymbol{Y}) \sim_s (\boldsymbol{X'}, \boldsymbol{Y'})$ *if, and only if,* $(\boldsymbol{X}_{\bullet i}\boldsymbol{Y}_{i\bullet})_{i=1}^r = (\boldsymbol{X'}_{\bullet i}\boldsymbol{Y'}_{i\bullet})_{i=1}^r$.

Proof sketch. The proof is deferred to the appendices. Indeed, the rank 1 contributions representation of $(\boldsymbol{X}, \boldsymbol{Y})$ and the one of $(\boldsymbol{X}\boldsymbol{D}, \boldsymbol{D}^{-1}\boldsymbol{Y})$ are the same for any diagonal matrix $\boldsymbol{D}$ with nonzero diagonal entries, because $\boldsymbol{X}_{\bullet i}\boldsymbol{Y}_{i\bullet} = (\boldsymbol{D}_{ii}\boldsymbol{X}_{\bullet i})(\frac{1}{\boldsymbol{D}_{ii}}\boldsymbol{Y}_{i\bullet})$ for all $i \in [\![r]\!]$. □

Remark. However, there is still a permutation ambiguity in this representation, because for any permutation matrix $\boldsymbol{P}$, the rank 1 contributions of $(\boldsymbol{X}, \boldsymbol{Y})$ and the one of $(\boldsymbol{X}\boldsymbol{P}, \boldsymbol{P}^{-1}\boldsymbol{Y})$ are not the same: they only differ by the order of their rank 1 contributions in the tuple.

When two pairs of factors are equivalent up to scaling and permutation, their rank 1 contributions representation are equivalent in the sense of the following lemma.

Lemma 4.4. ($\star$) *Let* $(\boldsymbol{X}, \boldsymbol{Y}), (\boldsymbol{X'}, \boldsymbol{Y'}) \in \mathbb{C}^{n\times r} \times \mathbb{C}^{r\times m}$. *Then,* $(\boldsymbol{X}, \boldsymbol{Y}) \sim (\boldsymbol{X'}, \boldsymbol{Y'})$ *if, and only if, there exists a permutation* $\sigma : [\![r]\!] \to [\![r]\!]$ *such that* $\boldsymbol{X'}_{\bullet i}\boldsymbol{Y'}_{i\bullet} = \boldsymbol{X}_{\bullet\sigma(i)}\boldsymbol{Y}_{\sigma(i)\bullet}$ *for all* $i \in [\![r]\!]$.

Proof sketch. The proof is straightforward and deferred to the appendices. □

We can also show below that there is an equivalence between the model sets $\prod_{i=1}^r \Sigma_{\boldsymbol{S_i}}$ and $\Sigma_{\hat{\boldsymbol{S}}}$.

Lemma 4.5. ($\star$) *Let* $\hat{\boldsymbol{S}} \in \mathbb{B}^{n\times r} \times \mathbb{B}^{r\times m}$. *Consider* $(\boldsymbol{A}, \boldsymbol{B}) \in \mathbb{C}^{n\times r} \times \mathbb{C}^{r\times m}$ *a pair of factors. Then,* $(\boldsymbol{A}_{\bullet i}\boldsymbol{B}_{i\bullet})_{i=1}^r \in \prod_{i=1}^r \Sigma_{\boldsymbol{S_i}}$ *if, and only if,* $(\boldsymbol{A}, \boldsymbol{B}) \in \Sigma_{\hat{\boldsymbol{S}}}$.

Remark. One can verify that the previous result is false when we replace non-extend model sets with extended model sets, *i.e.*, $\prod_{i=1}^r \Sigma_{\boldsymbol{S_i}}$ with $\prod_{i=1}^r \overline{\Sigma}_{\boldsymbol{S_i}}$ and $\Sigma_{\hat{\boldsymbol{S}}}$ with $\overline{\Sigma}_{\hat{\boldsymbol{S}}}$.

Proof. The proof is straightforward and deferred to the appendices. □

As we will see in the following sections, the rank 1 contributions representation plays an important role for the characterization of global fixed-support identifiability (Definition 4.1).

4.3 Fixing a pair of support with redundancy

In this section, we deal with the case where the pair of supports $\hat{\boldsymbol{S}}$ presents a redundant structure (Definition 4.2). In fact, Proposition 4.1 below shows that when $\hat{\boldsymbol{S}}$ has a redundancy, $\hat{\boldsymbol{S}}$ is not globally identifiable with fixed support (Definition 4.1). The proof of this result relies on the rank 1 contributions representation introduced in Section 4.2. We start with the following lemma which characterizes redundancy in a pair of supports by using the rank 1 contributions representation.

Lemma 4.6. *Let $\hat{\boldsymbol{S}} \in \mathbb{B}^{n\times r} \times \mathbb{B}^{r\times m}$ be a pair of supports. Then, $\hat{\boldsymbol{S}}$ has a redundant structure if, and only if, there exists $j_1, j_2 \in [\![r]\!]$ such that $j_1 \neq j_2$ and $\boldsymbol{S_{j_1}} = \boldsymbol{S_{j_2}}$.*

Remark. We recall that $\boldsymbol{S_{j_1}}$, $\boldsymbol{S_{j_2}}$ is the notation for the j_1-th and j_2-th rank 1 supports introduced in (4.3). Essentially, this lemma claims that $\hat{\boldsymbol{S}}$ has a redundancy if, and only if, at least two of its rank 1 supports are equal.

Proof. If $\hat{\boldsymbol{S}}$ has a redundant structure, we fix $j_1, j_2 \in [\![r]\!]$ such that $j_1 \neq j_2$ and $((\boldsymbol{S_L})_{\bullet j_1}, (\boldsymbol{S_R})_{j_1 \bullet}) = ((\boldsymbol{S_L})_{\bullet j_2}, (\boldsymbol{S_R})_{j_2 \bullet})$. Then, by definition given in (4.3), we obtain $\boldsymbol{S}_{\bullet j_1} = \boldsymbol{S}_{\bullet j_2}$. Conversely, let us consider $j_1, j_2 \in [\![r]\!]$ such that $j_1 \neq j_2$ and $\boldsymbol{S}_{\bullet j_1} = \boldsymbol{S}_{\bullet j_2}$. In other words, we have $(\boldsymbol{S_L})_{\bullet j_1} (\boldsymbol{S_R})_{j_1 \bullet} = (\boldsymbol{S_L})_{\bullet j_2} (\boldsymbol{S_R})_{j_2 \bullet}$, and by [15, Chapter 7, Lemma 1], we obtain $((\boldsymbol{S_L})_{\bullet j_1}, (\boldsymbol{S_R})_{j_1 \bullet}) = ((\boldsymbol{S_L})_{\bullet j_2}, (\boldsymbol{S_R})_{j_2 \bullet})$. ☐

Proposition 4.1. ($\star$) *Let $\hat{\boldsymbol{S}} \in \mathbb{B}^{n\times r} \times \mathbb{B}^{r\times m}$ be a pair of supports. If $\hat{\boldsymbol{S}}$ has a redundant structure, then $\hat{\boldsymbol{S}}$ is not globally identifiable with fixed support.*

Proof sketch. The idea of the proof is that when there exists two equal rank 1 supports $\boldsymbol{S_i} = \boldsymbol{S_j}$ for the pair of supports $\hat{\boldsymbol{S}}$, it is impossible to recover the rank 1 contributions corresponding to the indices i and j in the pair of factors, because the entries of the i-th contribution is completely "covered" by the j-th contribution, and vice versa. Indeed, values in the i-th contribution can be retrieved and then added in the j-th contribution, without changing the total sum, which leads to the same matrix product. This precisely means that the pair of supports is not globally identifiable with fixed support. The formal proof is deferred to the appendices. ☐

The idea of covering a rank 1 contribution by another one will be reused in the next section, when we consider the case where the pair of supports does not have a redundant structure.

4.4 Fixing a pair of supports without symmetry

Now, we suppose that the fixed pair of supports $\hat{\boldsymbol{S}}$ does not have a redundant structure (Definition 4.2). In other words, because of Lemma 4.2, we focus on the characterization of global fixed-support identifiability up to scaling (Definition 4.3).

4.4.1 Lifting

We use the lifting framework introduced in [9]. Define the lifting operator $\mathscr{S}_r^{n\times m} : (\mathbb{C}^{n\times m})^r \to \mathbb{C}^{n\times m}$ by:

$$\forall \underline{\boldsymbol{X}} \in (\mathbb{C}^{n\times m})^r, \quad \mathscr{S}_r^{n\times m}(\underline{\boldsymbol{X}}) := \sum_{i=1}^{r} \boldsymbol{X_i} \tag{4.6}$$

which returns the sum of r matrices of size $n \times m$. When there is no ambiguity, we simply denote this operator $\mathscr{S}$, or $\mathscr{S}_r$ when we want to precise only the number of matrices r, assuming that the size $n \times m$ is implicit. One important remark about the operator $\mathscr{S}$ is that for $(\boldsymbol{X}, \boldsymbol{Y}) \in \mathbb{C}^{n\times r} \times \mathbb{C}^{r\times m}$, we have:

$$\mathscr{S}((\boldsymbol{X}_{\bullet i} \boldsymbol{Y}_{i\bullet})_{i=1}^{r}) = \sum_{i=1}^{r} \boldsymbol{X}_{\bullet i} \boldsymbol{Y}_{i\bullet} = \boldsymbol{XY}. \tag{4.7}$$

This means that when manipulating the operator $\mathscr{S}$, we are implicitly working with the rank 1 contributions representation. We also remark that the operator $\mathscr{S}$ is invariant to permutation, in the sense that for all $\underline{\boldsymbol{X}} \in (\mathbb{C}^{n\times m})^r$, we have:

$$\forall \sigma : [\![r]\!] \to [\![r]\!] \text{ permutation, } \mathscr{S}(\underline{\boldsymbol{X}}) = \mathscr{S}((\boldsymbol{X}_{\boldsymbol{\sigma(i)}})_{i=1}^r). \tag{4.8}$$

4.4.2 Notations for model sets

In the following characterization, we will also use some specific model sets and secant sets. Let $\boldsymbol{S} \in \mathbb{B}^{n\times m} \cap \mathcal{R}_1$ be a rank 1 support. For any integer k, we denote the set of matrices of rank at most k (partially) supported by the support $\boldsymbol{S}$ as:

$$\Sigma_{\boldsymbol{S},k} := \Sigma_{\boldsymbol{S}} \cap \mathcal{R}_k, \tag{4.9}$$

$$\overline{\Sigma}_{\boldsymbol{S},k} := \overline{\Sigma}_{\boldsymbol{S}} \cap \mathcal{R}_k. \tag{4.10}$$

Then, we denote the following secant sets:

$$\Delta_{\boldsymbol{S},k} := \Sigma_{\boldsymbol{S},k} - \Sigma_{\boldsymbol{S},k}, \tag{4.11}$$

$$\overline{\Delta}_{\boldsymbol{S},k} := \overline{\Sigma}_{\boldsymbol{S},k} - \overline{\Sigma}_{\boldsymbol{S},k}. \tag{4.12}$$

4.4.3 Restricted rank 2 null space and identifiability

The null space of $\mathscr{S}$ is denoted $\mathcal{N}(\mathscr{S})$, and the set of r-tuples of matrices with rank at most k in the null space of $\mathscr{S}$ is denoted:

$$\mathcal{N}(\mathscr{S}, k) := \mathcal{N}(\mathscr{S}) \cap (\mathcal{R}_k)^r \tag{4.13}$$

which is also called in the following the *rank k null space* of $\mathscr{S}$. Now, Proposition 4.2 below characterizes global fixed-support identifiability up to scaling (Definition 4.3) for a pair of factors $\hat{\boldsymbol{S}}$ using the rank 2 null space of the linear application $\mathscr{S}$ intersected with a secant set. This proposition is actually inspired from [9], but instead of using the linear operator from [9], we use the linear operator $\mathscr{S}$.

Proposition 4.2. ($\star$) *Let $\hat{\boldsymbol{S}} \in \mathbb{B}^{n\times r} \times \mathbb{B}^{r\times m}$ be a pair of supports. Then, $\hat{\boldsymbol{S}}$ is globally identifiable with fixed support up to scaling if, and only if,:*

$$\mathcal{N}(\mathscr{S}) \cap \prod_{i=1}^{r} \Delta_{\boldsymbol{S_i},1} = \{0\}. \tag{4.14}$$

Proof sketch. The proof is deferred to the appendices.

□

One can verify that $\mathcal{N}(\mathscr{S}) \cap \prod_{i=1}^r \Delta_{\boldsymbol{S_i},1} \subseteq \mathcal{N}(\mathscr{S}, 2)$. In the following, a subset of $\mathcal{N}(\mathscr{S}, 2)$ will be usually called a *restricted rank 2 null space*. As a consequence of Proposition 4.2, in order to study global fixed-support identifiability up to scaling (Definition 4.3), we will focus on the characterization of (4.14), *i.e.*, the triviality of the restricted rank 2 null space $\mathcal{N}(\mathscr{S}) \cap \prod_{i=1}^r \Delta_{\boldsymbol{S_i},1}$. In practice, condition (4.14) can be hard to verify directly, so we will see in the next paragraphs how to simplify this condition.

Simplifying the expression of the restricted rank 2 null space? A natural approach to characterize condition (4.14) is to try to simplify the expression of the restricted rank 2 null space $\mathcal{N}(\mathscr{S}) \cap \prod_{i=1}^r \Delta_{\boldsymbol{S_i},1}$. Indeed, a natural question would be: do we have the equality $\mathcal{N}(\mathscr{S}) \cap \prod_{i=1}^r \Delta_{\boldsymbol{S_i},1} = \mathcal{N}(\mathscr{S}) \cap \prod_{i=1}^r \overline{\Sigma}_{\boldsymbol{S_i},2}$? Section A.2 in the appendices addresses this question. We show in Lemma A.8 that $\mathcal{N}(\mathscr{S}) \cap \prod_{i=1}^r \Delta_{\boldsymbol{S_i},1}$ is included in $\mathcal{N}(\mathscr{S}) \cap \prod_{i=1}^r \overline{\Sigma}_{\boldsymbol{S_i},2}$, but the two sets are not equal. However, Corollary A.3 shows that the two sets are "almost" equal, in a sense that for almost every $\underline{\boldsymbol{X}} \in \mathcal{N}(\mathscr{S}) \cap \prod_{i=1}^{r} \overline{\Sigma}_{\boldsymbol{S}_i}$ (for the Lebesgue measure), we have $\underline{\boldsymbol{X}} \in \mathcal{N}(\mathscr{S}) \cap \prod_{i=1}^{r} \Delta_{\boldsymbol{S}_i,1} \iff \underline{\boldsymbol{X}} \in \mathcal{N}(\mathscr{S}) \cap \prod_{i=1}^{r} \overline{\Sigma}_{\boldsymbol{S}_i,2}$. Since the two sets are not exactly equal, we prefer to directly characterize condition (4.14) in the following paragraphs, without simplifying the expression $\mathcal{N}(\mathscr{S}) \cap \prod_{i=1}^{r} \Delta_{\boldsymbol{S}_i,1}$.

4.4.4 Rank 1 matrix completability conditions

In this subsection, inspired by [15], we characterize the triviality of the restricted rank 2 null space in (4.14) using *rank 1 matrix completability*, which is the property formulating the idea that one can fill the missing values of a partially observed rank 1 matrix (see [14] for more details).

Rank 1 matrix completability definitions

We give a first naive definition of matrix completability in Definition 4.4.

Definition 4.4 (Completable rank 1 support, with zero)**.** Let $\boldsymbol{S}, \boldsymbol{S'} \in \mathbb{B}^{n\times m} \cap \mathcal{R}_1$ be two rank 1 supports such that $\boldsymbol{S'} \subseteq \boldsymbol{S}$. We say that $\boldsymbol{S}$ is completable (with zero) from $\boldsymbol{S'}$ if, for all $\boldsymbol{C}, \boldsymbol{D} \in \overline{\Sigma}_{\boldsymbol{S},1}$ verifying $\boldsymbol{C}_{|\boldsymbol{S'}} = \boldsymbol{D}_{|\boldsymbol{S'}}$, we have $\boldsymbol{C} = \boldsymbol{D}$.

However, as it is shown in Lemma 4.7, Definition 4.4 is too restrictive as a definition of matrix completability, because according to this definition, a rank 1 support $\boldsymbol{S}$ is completable if, and only if, we observe the whole support.

Lemma 4.7. *Let $\boldsymbol{S}, \boldsymbol{S'} \in \mathbb{B}^{n\times m} \cap \mathcal{R}_1$ be two rank 1 supports such that $\boldsymbol{S'} \subseteq \boldsymbol{S}$. Then, $\boldsymbol{S}$ is completable from $\boldsymbol{S'}$ in the sense of Definition 4.4 if, and only if, $\boldsymbol{S'} = \boldsymbol{S}$.*

Proof. Suppose $\boldsymbol{S'} \subsetneq \boldsymbol{S}$. Then, there exists $(k_0, l_0) \in \boldsymbol{S}$ such that $(k_0, l_0) \notin \boldsymbol{S'}$. Define the rank 1 matrices $\boldsymbol{C} := \boldsymbol{e}_{\boldsymbol{k_0}} \boldsymbol{e}_{\boldsymbol{l_0}}^T$ and $\boldsymbol{D} := 2\boldsymbol{e}_{\boldsymbol{k_0}} \boldsymbol{e}_{\boldsymbol{l_0}}^T$ (we can choose any values $\lambda \neq 1$ instead of 2). Then, by definition, for all $(k, l) \in \boldsymbol{S} \backslash \{(k_0, l_0)\}$, we have $\boldsymbol{C}_{kl} = 0 = \boldsymbol{D}_{kl}$. In particular, since $\boldsymbol{S'} \subseteq \boldsymbol{S} \backslash \{(k_0, l_0)\}$, we have $\boldsymbol{C}_{|\boldsymbol{S'}} = 0 = \boldsymbol{D}_{|\boldsymbol{S'}}$. But $\boldsymbol{C} \neq \boldsymbol{D}$, which shows that $\boldsymbol{S}$ is not completable from $\boldsymbol{S'}$ in the sense of Definition 4.4. □

Therefore, in the following, we will only use Definition 4.5 below as the correct definition of matrix completability. The main difference is that we do not allow zero entries on the rank 1 support $\boldsymbol{S}$. In other words, we consider the non-extended model set $\Sigma_{\boldsymbol{S},1}$ instead of the extended model set $\overline{\Sigma}_{\boldsymbol{S},1}$.

Definition 4.5 (Completable rank 1 support, without zero)**.** Let $\boldsymbol{S}, \boldsymbol{S'} \in \mathbb{B}^{n\times m} \cap \mathcal{R}_1$ be two rank 1 supports such that $\boldsymbol{S'} \subseteq \boldsymbol{S}$. We say that $\boldsymbol{S}$ is completable (without zero) from $\boldsymbol{S'}$ if, for all $\boldsymbol{C}, \boldsymbol{D} \in \Sigma_{\boldsymbol{S},1}$ verifying $\boldsymbol{C}_{|\boldsymbol{S'}} = \boldsymbol{D}_{|\boldsymbol{S'}}$, we have $\boldsymbol{C} = \boldsymbol{D}$.

Remark. We refer the reader to [15, Chapter 7, Lemma 2] for a characterization of completable rank 1 supports in the sense of Definition 4.5, by using bipartite graph representation. For instance, we can show that the rank 1 support $\boldsymbol{S}$ is completable from $\boldsymbol{S'}$, where $\boldsymbol{S}$ and $\boldsymbol{S'}$ are defined as:

$$\boldsymbol{S} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{S'} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.15}$$

Basically, in this example, for any matrix $\boldsymbol{M} \in \Sigma_{\boldsymbol{S},1}$ of rank at most 1 of the form:

$$\boldsymbol{M} = \begin{pmatrix} 0 & \star & ? & ? \\ 0 & \star & ? & ? \\ 0 & \star & \star & \star \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.16}$$

one can fill the missing values "?" without ambiguity by observing only the nonzero values ($\star$), with the constraint that $\boldsymbol{M}$ is at most of rank 1.

We highlight here a basic remark about matrix completability, that will be used in the next paragraphs.

Lemma 4.8. *Let $\boldsymbol{S} \in \mathbb{B}^{n\times m} \cap \mathcal{R}_1$ be a rank 1 support. Then $\boldsymbol{S}$ is completable from $\boldsymbol{S}$.*

Proof. This is a direct consequence of the definition of matrix completability. $\square$

We also want to highlight the following fact: when the observable support is in the form of $\boldsymbol{S}\backslash\boldsymbol{S'}$ for two given rank 1 supports $\boldsymbol{S}, \boldsymbol{S'} \in \mathbb{B}^{n\times m} \cap \mathcal{R}_1$, the following lemma gives an easy sufficient condition for completability of $\boldsymbol{S}$ from $\boldsymbol{S}\backslash\boldsymbol{S'}$.

Lemma 4.9. *Let $\boldsymbol{S}, \boldsymbol{S'} \in \mathbb{B}^{n\times m} \cap \mathcal{R}_1$ be two rank 1 supports. We denote $(\boldsymbol{s_L}, \boldsymbol{s_R}), (\boldsymbol{s'_L}, \boldsymbol{s'_R}) \in \mathbb{B}^n \times \mathbb{B}^m$ the unique vector supports such that $\boldsymbol{S} = \boldsymbol{s_L}\boldsymbol{s_R}^T$ and $\boldsymbol{S'} = \boldsymbol{s'_L}\boldsymbol{s'_R}^T$. Suppose that $\boldsymbol{s_L} \not\subseteq \boldsymbol{s'_L}$ and $\boldsymbol{s_R} \not\subseteq \boldsymbol{s'_R}$. Then, $\boldsymbol{S}$ is completable from $\boldsymbol{S}\backslash\boldsymbol{S'}$.*

Proof. Let $\boldsymbol{C}, \boldsymbol{D} \in \Sigma_{\boldsymbol{S},1}$ such that $\boldsymbol{C}_{|\boldsymbol{S}\backslash\boldsymbol{S'}} = \boldsymbol{D}_{|\boldsymbol{S}\backslash\boldsymbol{S'}}$. By assumption, there exists a row index $k_0 \in \boldsymbol{s_L}\backslash\boldsymbol{s'_L}$ and a column index $l_0 \in \boldsymbol{s_R}\backslash\boldsymbol{s'_R}$. Then, for all $k \in \boldsymbol{s_L}$, we have $(k, l_0) \in \boldsymbol{S}\backslash\boldsymbol{S'}$, which means that $\boldsymbol{C}_{kl_0} = \boldsymbol{D}_{kl_0}$. And for all $l \in \boldsymbol{s_R}$, we have $(k_0, l) \in \boldsymbol{S}\backslash\boldsymbol{S'}$, which means that $\boldsymbol{C}_{k_0l} = \boldsymbol{D}_{k_0l}$. This means that $\boldsymbol{C}_{k_0\bullet} = \boldsymbol{D}_{k_0\bullet}$ and $\boldsymbol{C}_{\bullet l_0} = \boldsymbol{D}_{\bullet l_0}$. Let $l \in \boldsymbol{s_R}$ be a column index. Since $\boldsymbol{C}, \boldsymbol{D} \in \Sigma_{\boldsymbol{S},1}$, we have:

$$\boldsymbol{C}_{\bullet l} = \frac{\boldsymbol{C}_{k_0l}}{\boldsymbol{C}_{k_0l_0}}\boldsymbol{C}_{\bullet l_0} = \frac{\boldsymbol{D}_{k_0l}}{\boldsymbol{D}_{k_0l_0}}\boldsymbol{D}_{\bullet l_0} = \boldsymbol{D}_{\bullet l}. \tag{4.17}$$

This is true for all $l \in s_R$, so we obtain $\boldsymbol{C} = \boldsymbol{D}$. $\square$

Iterative completability from observable supports

Based on the notion of matrix completability introduced in the previous paragraph, we now want to define in Definition 4.6 below the property that, given $\underline{\boldsymbol{S}}$ a r-tuple of rank 1 supports, we can complete one by one the rank 1 supports $\boldsymbol{S_i}$ from the indices of $\boldsymbol{S_i}$ not covered by the other rank 1 supports $(\boldsymbol{S_{i'}})_{i'\neq i}$. Let $(\boldsymbol{S_i})_{i\in I} \in (\mathbb{B}^{n\times m} \cap \mathcal{R}_1)^I$ be a tuple of rank 1 supports indexed by I. The *observable support* indexed by $i \in I$ in the tuple $(\boldsymbol{S_i})_{i\in I}$ is defined as:

$$(\boldsymbol{S_i})^I := \boldsymbol{S_i}\backslash \bigcup_{i'\in I\backslash\{i\}} \boldsymbol{S_{i'}}. \tag{4.18}$$

Remark. We draw the reader's attention to this notation. Let $(\boldsymbol{S_i})_{i\in I} \in (\mathbb{B}^{n\times m} \cap \mathcal{R}_1)^I$ be a tuple of rank 1 supports. For a given $i \in I$, the observable support $(\boldsymbol{S_i})^I$ defined in (4.18) depends on $(\boldsymbol{S_{i'}})_{i'\in I}$ and not only on $\boldsymbol{S_i}$. Moreover, the set $(\boldsymbol{S_i})^J$ is decreasing with respect to the size of a subset $J \subseteq I$, in the sense that for two subsets $J, J' \subseteq I$ such that $J' \subseteq J$, and for any index $i \in J'$, we have $(\boldsymbol{S_i})^J \subseteq (\boldsymbol{S_i})^{J'}$. For these reasons, one might consider in a future work more satisfying notations for observable supports than the one proposed in this work.

Definition 4.6 (Iterative completability from observable supports)**.** Let $\underline{\boldsymbol{S}} \in (\mathbb{B}^{n\times m} \cap \mathcal{R}_1)^r$ be a r-tuple of rank 1 supports. We say that $\underline{\boldsymbol{S}}$ is iteratively completable from observable supports if there exists a permutation $\sigma : [\![r]\!] \to [\![r]\!]$ such that for all $i \in [\![r]\!]$, $\boldsymbol{S_{\sigma(i)}}$ is completable from the observable support $(\boldsymbol{S_{\sigma(i)}})^{\sigma([\![i;r]\!])}$.

Based on rank 1 matrix completability conditions given in [15, Chapter 7, Lemma 2], we give some examples of r-tuples of rank 1 supports which are (not) iteratively completable from observable supports in Figure 4.2. One class of iteratively completable r-tuples of rank 1 supports is the one where the rank 1 supports are all disjoint, as it is shown in the following lemma.

$$\begin{pmatrix} 0 & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \star & \star & 0 \\ \star & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{pmatrix} \quad \begin{pmatrix} \star & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{pmatrix}$$

(a) Iteratively completable

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ \star & \star & \star & \star \\ \star & \star & \star & \star \\ 0 & 0 & 0 & \star \end{pmatrix} \quad \begin{pmatrix} 0 & \star & \star & 0 \\ \star & \star & \star & \star \\ \star & \star & \star & \star \\ 0 & \star & \star & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(b) Not iterativeley completable

Figure 4.2: Three examples of 2-tuples of rank 1 supports which are iteratively completable from observable supports, and three other which are not. In each example, the first (resp. second) rank 1 support in the tuple is represented in a red (resp. green) rectangle: indices inside the rectangle correspond to indices with a value 1, while indices outside the rectangle correspond to indices with a value 0. Nonzero values are denoted by a star symbol ($\star$).

Lemma 4.10. *Let $\underline{\boldsymbol{S}} \in (\mathbb{B}^{n\times m} \cap \mathcal{R}_1)^r$ be a r-tuple of rank 1 supports. Suppose that the rank 1 supports $(\boldsymbol{S_i})_{i=1}^r$ are disjoint,* i.e., *for all $i, j \in [\![r]\!]$ such that $i \neq j$, we have $\boldsymbol{S_i} \cap \boldsymbol{S_j} = \emptyset$. Then, $\hat{\boldsymbol{S}}$ is iteratively completable from observable supports.*

Proof. Let $i \in [\![r]\!]$. Then, the observable support $(\boldsymbol{S_i})^{[\![i;r]\!]}$ is equal to

$$\boldsymbol{S_i} \backslash \bigcup_{i' \in [\![i;r]\!] \backslash \{i\}} (\boldsymbol{S_{i'}} \cap \boldsymbol{S_i}) = \boldsymbol{S_i}, \tag{4.19}$$

which means that $\boldsymbol{S_i}$ is completable from $\boldsymbol{S_i} = (\boldsymbol{S_i})^{[\![i;r]\!]}$ by Lemma 4.8. This is true for any $i \in [\![r]\!]$, so by definition $\underline{\boldsymbol{S}}$ is iteratively completable from observable supports. $\square$

Remark. This lemma seems trivial, but [15, Chapter 7, Section 7.4] shows some non-trivial consequences of this lemma, since it is used to prove that the butterfly factorization [18, 11] in two factors of the discrete Fourier transform matrix is identifiable [15, Chapter 7, Lemma 5], when the left factor is $\frac{N}{2}$-sparse by column (at most $\frac{N}{2}$ nonzero entries per column) and the right factor is 2-sparse by row (at most 2 nonzero entries per row), where N is the size of the DFT matrix.

4.4.5 Characterization with iterative completability

The idea behind iterative completability from observable supports is that we can complete one by one the rank 1 contributions of a pair of factors, by observing only its entries on its observable support, which is the subsupport not covered by the other rank 1 supports. Once a rank 1 contribution is completed, we put aside its contribution and focus on the remaining rank 1 contributions. In fact, we will show that iterative completability from observable supports is a sufficient condition for having a trivial restricted rank 2 null space, and a necessary condition for the case $r = 2$ (when considering two columns for the left factor, and two rows for the right factor).

Iterative completability as a sufficient condition

Proposition 4.3 below shows that iterative completability from observable supports is a sufficient condition for having a trivial restricted rank 2 null space. The main idea is that when $\underline{\boldsymbol{S}}$ is iteratively completable from observable supports, we can complete the rank 1 supports from their observable supports one by one, in the sense of the following lemma.

Lemma 4.11. *Let I be a subset of indices, $i \in I$ be an index, and $\underline{\boldsymbol{S}} \in (\mathbb{B}^{n\times m} \cap \mathcal{R}_1)^I$ be a tuple of rank 1 supports indexed by I. Suppose that $\boldsymbol{S_i}$ is completable from the observable support $(\boldsymbol{S_i})^I$. Then, for all $\underline{\boldsymbol{X}} \in \prod_{i\in I} \Delta_{\boldsymbol{S_i},1}$ such that $\sum_{i'\in I} \boldsymbol{X_{i'}} = 0$, we have $\boldsymbol{X_i} = 0$.*

Proof. Let $\underline{\boldsymbol{X}} \in \prod_{i\in I} \Delta_{\boldsymbol{S_i},1}$ such that $\sum_{i'\in I} \boldsymbol{X_{i'}} = 0$. Since $\boldsymbol{X_i} \in \Delta_{\boldsymbol{S_i},1} = \Sigma_{\boldsymbol{S_i},1} - \Sigma_{\boldsymbol{S_i},1}$, there exists $\boldsymbol{C}, \boldsymbol{D} \in \Sigma_{\boldsymbol{S_i}}$ such that $\boldsymbol{X_i} = \boldsymbol{C} - \boldsymbol{D}$. Then, since $\sum_{i\in I} \boldsymbol{X_i} = 0$, we have $0 = (\boldsymbol{X_i})_{kl} = \boldsymbol{C}_{kl} - \boldsymbol{D}_{kl}$ for all $(k,l) \in (\boldsymbol{S_i})^I$. But $\boldsymbol{S_i}$ is completable from $(\boldsymbol{S_i})^I$, so by definition, $\boldsymbol{C} = \boldsymbol{D}$, which leads to $\boldsymbol{X_i} = 0$. □

Proposition 4.3. *Let $\underline{\boldsymbol{S}} \in (\mathbb{B}^{n\times m} \cap \mathcal{R}_1)^r$ be a r-tuple of rank 1 supports. If $\underline{\boldsymbol{S}}$ is iteratively completable from observable supports, then we have $\mathcal{N}(\mathscr{S}) \cap \prod_{i=1}^r \Delta_{\boldsymbol{S_i},1} = \{0\}$.*

Proof. Let $\underline{\boldsymbol{X}} \in \mathcal{N}(\mathscr{S}) \cap \prod_{i=1}^r \Delta_{\boldsymbol{S_i},1}$. We fix a permutation $\sigma : [\![r]\!] \to [\![r]\!]$ such that for all $i \in [\![r]\!]$, $\boldsymbol{S_{\sigma(i)}}$ is completable from the observable support $(\boldsymbol{S_{\sigma(i)}})^{\sigma([\![i;r]\!])}$. In particular, this means, by Lemma 4.11, that $\boldsymbol{X_{\sigma(1)}} = 0$. Then, let $k \in [\![r-1]\!]$, and suppose that for all $j \in [\![k]\!]$, $\boldsymbol{X_{\sigma(j)}} = 0$. We want to show $\boldsymbol{X_{\sigma(k+1)}} = 0$. Since $\underline{\boldsymbol{X}} \in \mathcal{N}(\mathscr{S})$, we have $0 = \sum_{i=1}^r \boldsymbol{X_i} = \sum_{j=1}^r \boldsymbol{X_{\sigma(j)}} = \sum_{j=k+1}^r \boldsymbol{X_{\sigma(j)}}$. But by assumption, $\boldsymbol{S_{\sigma(k+1)}}$ is completable from the observable support $\left(\boldsymbol{S_{\sigma(k+1)}}\right)^{\sigma([\![k+1;r]\!])}$. This means, by Lemma 4.11, that $\boldsymbol{X_{\sigma(k+1)}} = 0$. In conclusion, for all $j \in [\![r]\!]$, we have $\boldsymbol{X_{\sigma(j)}} = 0$, which means that $\underline{\boldsymbol{X}} = 0$. □

Iterative completability as a necessary condition

For the case $r = 2$, the converse of Proposition 4.3 is actually true. Indeed, the case $r = 2$ is a simple case where iterative completability from observable supports can be reduced to rank 1 matrix completability of one of the two rank 1 supports from its observable support, as it is shown in the following lemma.

Lemma 4.12. *Let $\underline{\boldsymbol{S}} \in (\mathbb{B}^{n\times m} \cap \mathcal{R}_1)^2$ be a pair of rank 1 supports. Then, $\underline{\boldsymbol{S}}$ is iteratively completable from observable supports if, and only if, there exists $i \in [\![2]\!]$ such that $\boldsymbol{S_i}$ is completable from $(\boldsymbol{S_i})^{[\![2]\!]}$.*

Proof. Suppopse there exists $i \in [\![2]\!]$ such that $\boldsymbol{S_i}$ is completable from $(\boldsymbol{S_i})^{[\![2]\!]}$. By Lemma 4.8, $\boldsymbol{S_j}$ is completable from $\boldsymbol{S_j}$, where we denoted $j \in [\![2]\!]\backslash\{i\}$ the unique element in $[\![2]\!]\backslash\{i\}$. Denote $\sigma : [\![2]\!] \to [\![2]\!]$ the permutation verifying $\sigma(1) = i$ and $\sigma(2) = j$. This gives the permutation in Definition 4.6 showing that $\underline{\boldsymbol{S}}$ is iteratively completable from observable supports. The converse is true by definition of iterative completability from observable supports in Definition 4.6. □

Finally, Proposition 4.4 below shows that iterative completability from observable supports is a necessary condition for having a trivial restricted rank 2 null space, in the case $r = 2$.

Proposition 4.4. ($\star$) *Let $\underline{\boldsymbol{S}} \in (\mathbb{B}^{n\times m} \cap \mathcal{R}_1)^2$ be a pair of rank 1 supports. If $\mathcal{N}(\mathscr{S}) \cap \prod_{i=1}^2 \Delta_{\boldsymbol{S_i},1} = \{0\}$, then $\underline{\boldsymbol{S}}$ is iteratively completable from observable supports.*

Proof sketch. Since there are only two rank 1 supports, the observable supports are the relative complements of a rank 1 support with respect to another rank 1 support. Then, we can use the easy sufficient condition given by Lemma 4.9 for completing a rank 1 support $\boldsymbol{S}$ from $\boldsymbol{S}\backslash\boldsymbol{S'}$ where $\boldsymbol{S'}$ is another rank 1 support. The idea then is to construct $i \in [\![2]\!]$ and $\boldsymbol{X_i} \in \Delta_{\boldsymbol{S_i},1}$ such that $\boldsymbol{X_i} \neq 0$ but $(\boldsymbol{X_i})_{\boldsymbol{S_i}\backslash\boldsymbol{S_j}} = 0$, where $j \in [\![2]\!]\backslash\{i\}$ is the unique element in $[\![2]\!]\backslash\{i\}$. We also want to construct $\boldsymbol{X_j} \in \Delta_{\boldsymbol{S_j},1}$ such that $(\boldsymbol{X_j})_{|\boldsymbol{S_j}\backslash\boldsymbol{S_i}} = 0$ and $(\boldsymbol{X_j})_{|\boldsymbol{S_j}\cap\boldsymbol{S_i}} = -(\boldsymbol{X_i})_{|\boldsymbol{S_i}\cap\boldsymbol{S_j}}$. Then, by construction, we would have $\boldsymbol{X_j} = -\boldsymbol{X_i}$, and $\underline{\boldsymbol{X}} \in \mathcal{N}(\mathscr{S}) \cap \prod_{i=1}^2 \Delta_{\boldsymbol{S_i},1}$ with $\underline{\boldsymbol{X}} \neq 0$. The formal proof is deferred to the appendices. □

With the previous paragraphs, we conclude that iterative completability from observable supports is a sufficient condition for global fixed-support identifiability up to scaling (Definition 4.3), and also a necessary condition for the case $r = 2$.

Theorem 4.1. *Let $\underline{\boldsymbol{S}} \in (\mathbb{B}^{n\times m} \cap \mathcal{R}_1)^2$ be a pair of rank 1 supports. Then, $\underline{\boldsymbol{S}}$ is globally identifiable with fixed support up to scaling (Definition 4.3) if, and only if, $\underline{\boldsymbol{S}}$ is iteratively completable from observable supports.*

Proof. We use Proposition 4.2, Proposition 4.3 and Proposition 4.4. □

4.4.6 Iterative partial completability

However, iterative completability from observable support is not a necessary condition for having a trivial restricted rank 2 null space in the case where $r \geq 3$, as it is shown in the following counter-example. In other words, the extension of Proposition 4.4 to $r \geq 3$ is false.

Example 4.1. Consider the following rank 1 supports for matrices of size 4×4:

$$\boldsymbol{S_1} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{S_2} := \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{S_3} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}. \tag{4.20}$$

Then, for all $i \in \llbracket 3 \rrbracket$, the support $\boldsymbol{S_i}$ is not completable from $(\boldsymbol{S_i})^{\llbracket 3 \rrbracket}$. Indeed, we have:

$$(\boldsymbol{S_1})^{\llbracket 3 \rrbracket} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad (\boldsymbol{S_2})^{\llbracket 3 \rrbracket} := \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (\boldsymbol{S_3})^{\llbracket 3 \rrbracket} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

and we can find:

$$\tilde{\boldsymbol{X}}_{\boldsymbol{1}} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \neq 0, \tag{4.21}$$

$$\tilde{\boldsymbol{X}}_{\boldsymbol{2}} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq 0, \tag{4.22}$$

$$\tilde{\boldsymbol{X}}_{\boldsymbol{3}} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 2 & 1 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \neq 0. \tag{4.23}$$

For each $i \in \llbracket 3 \rrbracket$, we have $\tilde{\boldsymbol{X}}_{\boldsymbol{i}} \in \Delta_{\boldsymbol{S_i},1}$, and $(\tilde{\boldsymbol{X}}_{\boldsymbol{i}})_{|(\boldsymbol{S_i})^{\llbracket 3 \rrbracket}} = 0$, but $\tilde{\boldsymbol{X}}_{\boldsymbol{i}} \neq 0$, which shows that $\boldsymbol{S_i}$ is not completable from $(\boldsymbol{S_i})^{\llbracket 3 \rrbracket}$. However, we can show that $\mathcal{N}(\mathscr{S}) \cap \prod_{i=1}^{3} \Delta_{\boldsymbol{S_i},1} = \{0\}$. Let $\underline{\boldsymbol{X}} \in \mathcal{N}(\mathscr{S}) \cap \prod_{i=1}^{3} \Delta_{\boldsymbol{S_i},1}$. We proceed in 5 steps.

1. Since $\sum_{i=1}^{3} \boldsymbol{X_i} = 0$, we have $(\boldsymbol{X_i})_{|\boldsymbol{S_i}^{\llbracket 3 \rrbracket}} = 0$ for each $i \in \llbracket 3 \rrbracket$. Indeed, for $i \in \llbracket 3 \rrbracket$, for $(k,l) \in (\boldsymbol{S_i})^{\llbracket 3 \rrbracket}$, we have: $0 = (\boldsymbol{X_i})_{kl} + (\boldsymbol{X_{i'}})_{kl} + (\boldsymbol{X_{i''}})_{kl}$ where i', i'' are the two remaining indices in $\llbracket 3 \rrbracket \backslash \{i\}$. But since $(k,l) \in (\boldsymbol{S_i})^{\llbracket 3 \rrbracket} = \boldsymbol{S_i} \backslash (\boldsymbol{S_{i'}} \cup \boldsymbol{S_{i''}})$, we have $0 = (\boldsymbol{X_{i'}})_{kl} = (\boldsymbol{X_{i''}})_{kl}$, which leads to $0 = (\boldsymbol{X_i})_{kl} + 0 + 0 = (\boldsymbol{X_i})_{kl}$.

2. Consider then the following supports:

$$\tilde{\boldsymbol{S}_2} := \begin{pmatrix} 0 & \boxed{\begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \end{pmatrix}, \quad \tilde{\boldsymbol{S}_2} \cap (\boldsymbol{S_2})^{[\![3]\!]} = \begin{pmatrix} 0 & \boxed{\begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \end{pmatrix}, \tag{4.24}$$

where we use the abuse of notation $\tilde{\boldsymbol{S}_2} \cap (\boldsymbol{S_2})^{[\![3]\!]}$ represented as a binary matrix. Since $\boldsymbol{X_2} \in \Delta_{\boldsymbol{S_2},1}$, we denote $\boldsymbol{C_2}, \boldsymbol{D_2} \in \Sigma_{\boldsymbol{S_2},1}$ such that $\boldsymbol{X_2} = \boldsymbol{C_2} - \boldsymbol{D_2}$. Define now $\tilde{\boldsymbol{C}_2}, \tilde{\boldsymbol{D}_2} \in \Sigma_{\tilde{\boldsymbol{S}_2},1}$ such that $(\tilde{\boldsymbol{C}_2})_{|\tilde{\boldsymbol{S}_2}} = (\boldsymbol{C_2})_{|\tilde{\boldsymbol{S}_2}}$, and $(\tilde{\boldsymbol{D}_2})_{|\tilde{\boldsymbol{S}_2}} = (\boldsymbol{D_2})_{|\tilde{\boldsymbol{S}_2}}$, which are the extractions of rank 1 matrices $\boldsymbol{C_2}, \boldsymbol{D_2}$ on the rank 1 support $\tilde{\boldsymbol{S}_2}$, completed with zero entries on the complement of the support $\tilde{\boldsymbol{S}_2}$. Since $\tilde{\boldsymbol{S}_2} \cap (\boldsymbol{S_2})^{[\![3]\!]} \subseteq \tilde{\boldsymbol{S}_2}$, we have:

$$\begin{aligned} (\tilde{\boldsymbol{C}_2})_{|\tilde{\boldsymbol{S}_2} \cap (\boldsymbol{S_2})^{[\![3]\!]}} - (\tilde{\boldsymbol{D}_2})_{|\tilde{\boldsymbol{S}_2} \cap (\boldsymbol{S_2})^{[\![3]\!]}} &= (\boldsymbol{C_2})_{|\tilde{\boldsymbol{S}_2} \cap (\boldsymbol{S_2})^{[\![3]\!]}} - (\boldsymbol{D_2})_{|\tilde{\boldsymbol{S}_2} \cap (\boldsymbol{S_2})^{[\![3]\!]}} \\ &= (\boldsymbol{C_2} - \boldsymbol{D_2})_{|\tilde{\boldsymbol{S}_2} \cap (\boldsymbol{S_2})^{[\![3]\!]}} \\ &= (\boldsymbol{X_2})_{|\tilde{\boldsymbol{S}_2} \cap (\boldsymbol{S_2})^{[\![3]\!]}} \\ &= 0. \end{aligned} \tag{4.25}$$

But we remark that the support $\tilde{\boldsymbol{S}_2}$ is completable from $\tilde{\boldsymbol{S}_2} \cap (\boldsymbol{S_2})^{[\![3]\!]}$, which means that $\tilde{\boldsymbol{C}_2} = \tilde{\boldsymbol{D}_2}$, and by definition, $(\boldsymbol{X_2})_{|\tilde{\boldsymbol{S}_2}} = 0$. Therefore, in particular $(\boldsymbol{X_2})_{22} = 0$.

3. Then, $(\boldsymbol{X_1})_{22} = 0 - (\boldsymbol{X_2})_{22} - (\boldsymbol{X_3})_{22} = 0$, because $(\boldsymbol{X_2})_{22} = 0$ from step 2 and $(\boldsymbol{X_3})_{22} = 0$ since $(2,2) \notin \mathrm{supp}(\boldsymbol{X_3})$. Combining this result with step 1, we have $(\boldsymbol{X_1})_{|(\boldsymbol{S_1})^{[\![3]\!]} \cup \{(2,2)\}} = 0$. But $\boldsymbol{S_1}$ is completable from $(\boldsymbol{S_1})^{[\![3]\!]} \cup \{(2,2)\}$, so $\boldsymbol{X_1} = 0$.

4. Then, $(\boldsymbol{X_3})_{42} = 0 - (\boldsymbol{X_1})_{42} - (\boldsymbol{X_2})_{42} = 0$, because $\boldsymbol{X_1} = 0$ from step 3, and $(\boldsymbol{X_2})_{42} = 0$ since $(4,2) \notin \mathrm{supp}(\boldsymbol{X_2})$. Combining this result with step 1, we have $(\boldsymbol{X_3})_{|(\boldsymbol{S_3})^{[\![3]\!]} \cup \{(4,2)\}} = 0$. But $\boldsymbol{S_3}$ is completable from $(\boldsymbol{S_3})^{[\![3]\!]} \cup \{(4,2)\}$, so $\boldsymbol{X_3} = 0$.

5. In conclusion, $\boldsymbol{X_2} = 0 - \boldsymbol{X_1} - \boldsymbol{X_3} = 0$, which means that $\underline{\boldsymbol{X}} = 0$.

In Example 4.1, we see that there is a more subtle notion of rank 1 matrix completability than iterative completability from observable supports. Indeed, the support $\boldsymbol{S_2}$ is not completable from $(\boldsymbol{S_2})^{[\![3]\!]}$, but it is *partially* completable, in the sense that the rank 1 subsupport $\tilde{\boldsymbol{S}_2} \subseteq \boldsymbol{S_2}$ is completable from $\tilde{\boldsymbol{S}_2} \cap (\boldsymbol{S_2})^{[\![3]\!]}$. Therefore, we suggest below a definition for the notion of *partial completability* of a rank 1 support $\boldsymbol{S} \in \mathbb{B}^{n \times m} \cap \mathcal{R}_1$ from an observable support $\boldsymbol{S'} \subseteq \boldsymbol{S}$.

Definition 4.7 (Partially completable rank 1 support)**.** Let $\boldsymbol{S}, \boldsymbol{S'} \in \mathbb{B}^{n \times m} \cap \mathcal{R}_1$ be two rank 1 supports such that $\boldsymbol{S'} \subseteq \boldsymbol{S}$. We say that $\boldsymbol{S}$ is partially completable from $\boldsymbol{S'}$ if there exists a rank 1 support $\tilde{\boldsymbol{S}} \subseteq \boldsymbol{S}$ such that $\tilde{\boldsymbol{S}}$ is not included in $\boldsymbol{S'}$, and $\tilde{\boldsymbol{S}}$ is completable from $\tilde{\boldsymbol{S}} \cap \boldsymbol{S'}$.

Remark. We require that $\tilde{\boldsymbol{S}}$ is not included in $\boldsymbol{S'}$ so that $\tilde{\boldsymbol{S}} \backslash \boldsymbol{S'} \neq \emptyset$. Then, completing $\tilde{\boldsymbol{S}}$ from $\tilde{\boldsymbol{S}} \cap \boldsymbol{S'}$ means that the entries indexed by $\tilde{\boldsymbol{S}} \backslash \boldsymbol{S'}$, which are not in the observed support $\boldsymbol{S'}$, can be completed.

Based on this partial completability property, one can extend the notion of iterative completability from observable support (Definition 4.6) to a more subtle notion of iterative *partial* completability. This new notion could then be a necessary and sufficient condition for a trivial restricted rank 2 null space for any case of r, and therefore for global fixed-support identifiability up to scaling (Definition 4.3). We leave this construction and this conjecture as a future work.

Chapter 5

Conclusion

In conclusion, we present a summary of some open questions suggested by this work, a synthesis of our analysis, and some possible impacts of this work.

5.1 Open questions

The following questions are left for possible future works.

1. What is the characterization of a left factor $\boldsymbol{X}$ which is not invariant to scaled permutations for a given family of allowed right supports (Definition 2.6)?

2. What is the characterization of global right identifiability (Definition 3.1) when the left factor $\boldsymbol{X}$ is degenerate for a given family of allowed right supports, in the sense that it is invariant to scaled permutations (Definition 2.6)?

3. Is iterative partial completability introduced in Section 4.4.6 a necessary and sufficient condition for global fixed-support identifiability (Definition 4.1)?

4. Based on the characterization of global identifiability when fixing the left factor or when fixing the pair of supports (Theorem 3.1, Theorem A.1, Theorem 4.1), how can we characterize global generic identifiability (Definition 2.4)?

5. After understanding how to characterize global identifiability, how can we characterize instance identifiability?

5.2 Synthesis

We have seen through the analysis of identifiability presented in Chapter 2 that right identifiability (Definition 2.5) and fixed-support identifiability (Definition 2.8) are necessary conditions for generic identifiability (Definition 2.3). Then, Chapter 3 and Chapter 4 gave some characterization of these necessary conditions, with Proposition 3.1, Theorem 3.1 for extended global exact right identifiability (Definition 3.3), and Proposition 4.2, Theorem 4.1 for global fixed-support identifiability up to scaling (Definition 4.3). In both problem variations, the idea was to characterize identifiability using triviality of some linear operator's null space intersected with specific secant sets determined by the family of allowed (pairs of) supports.

- When fixing the left factor $\boldsymbol{X}$, given a family of allowed right supports Ω, the linear operator was represented by the matrix $(\boldsymbol{I_m} \otimes \boldsymbol{X})$, and the secant sets where $\overline{\Sigma}_{\boldsymbol{S_R}} - \overline{\Sigma}_{\boldsymbol{S_R}'}$ for each pairs of allowed right supports $(\boldsymbol{S_R}, \boldsymbol{S_R}')$. This gave condition 2 in Proposition 3.1.

- When fixing the pair of supports $\hat{\boldsymbol{S}}$, the linear operator was $\mathscr{S} : (\boldsymbol{X_i})_{i=1}^r \mapsto \sum_{i=1}^r \boldsymbol{X_i}$, and the secant set was $\prod_{i=1}^r \Sigma_{\boldsymbol{S_i},1} - \Sigma_{\boldsymbol{S_i},1}$. This gave condition 2 in Proposition 4.2.

This kind of characterization seems to be unavoidable when addressing identifiability issues in matrix sparse factorization.

5.3 Possible impact and future direction

We propose here some future research directions based on this work.

General characterization of identifiability Because it considers only specific variations of the generic problem of identifiability, this work is only a first step to construct general characterizations of identifiability. The synthesis proposed in the previous paragraph might be a first step for this kind of future work. We hope that such characterization will help us to better understand the difficulties behind matrix sparse factorization, and how to get around these obstacles with adapted algorithms.

Algorithm for matrix sparse factorization This work might also suggests some ideas for designing algorithms for matrix sparse factorization. Indeed, when fixing the pair of supports, the idea of (partial) iterative completability can lead to a greedy approach where the algorithm completes the rank 1 contributions one by one in order to recover the sparse factors.

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Appendix A

Important extensions

As explained in Section 1.5, we present, in this chapter, some important extensions proposed by the main text.

A.1 General family of allowed right supports

This extension was proposed in Section 3.3 of the main text. Hopefully, we obtain at the end, with Theorem A.1 below, a general characterization of global exact right identifiability (Definition 3.2) which can be applied to any family of allowed right factors (not necessarily a closed family). The obtained characterization can then be reduced to condition 5 of Theorem 3.1 when considering the specific case of a closed family of right supports.

A.1.1 Characterization of global exact right identifiability

We start this general characterization similarly to Section 3.2.1. We use the vectorization operator in (3.1) and Lemma 3.2 to prove Proposition A.1 below which is inspired by Proposition 3.1.

Proposition A.1. *Let $\Omega_R \subseteq \mathbb{B}^{r\times m}$ be a family of allowed right supports, and $\boldsymbol{X} \in \mathbb{C}^{n\times r}$ a fixed left factor. Then the following assertions are equivalent:*

1. *$(\Omega_R, \boldsymbol{X})$ is globally and exactly right identifiable (Definition 3.2);*
2. *for all $\boldsymbol{S_R}, \boldsymbol{S_R}' \in \Omega_R$, we have $\operatorname{Ker}(\boldsymbol{I_m} \otimes \boldsymbol{X}) \cap \left(\Sigma_{\operatorname{vec}(\boldsymbol{S_R})} - \Sigma_{\operatorname{vec}(\boldsymbol{S_R}')}\right) \subseteq \{0\}$.*

Proof. The proof is the same as the proof of Proposition 3.1 given in the main text, except that we replace extended model sets $\overline{\Sigma}_{\bullet}$ by non-extended model sets $\Sigma_{\bullet}$. □

Remark. We can prove Proposition 3.1 as a corollary of this proposition by considering the specific case where $\Omega_R = \overline{\Omega}_R$. This proof is done in the next chapter.

We now want to give a more intuitive characterization of global exact right identifiability (Definition 3.2) than condition 2 of Proposition A.1. By using the expression of the secant set $\Sigma_{\operatorname{vec}(\boldsymbol{S_R})} - \Sigma_{\operatorname{vec}(\boldsymbol{S_R})}$ given by the second assertion of Lemma 3.3, we obtain the following lemma.

Lemma A.1. *Let $\boldsymbol{S_R}, \boldsymbol{S_R}' \in \mathbb{B}^{r\times m}$ be two right supports, and $\boldsymbol{X} \in \mathbb{C}^{n\times r}$ a fixed left factor. Then, the following assertions are equivalent:*

1. *$\operatorname{Ker}(\boldsymbol{I_m} \otimes \boldsymbol{X}) \cap \left(\Sigma_{\operatorname{vec}(\boldsymbol{S_R})} - \Sigma_{\operatorname{vec}(\boldsymbol{S_R}')}\right) \subseteq \{0\}$;*
2. *$\operatorname{Ker}(\boldsymbol{I_m} \otimes \boldsymbol{X}) \cap \overline{\Sigma}_{\operatorname{vec}(\boldsymbol{S_R})\cup\operatorname{vec}(\boldsymbol{S_R}')} \subseteq \{0\} \cup \bigcup_{i\in\operatorname{vec}(\boldsymbol{S_R})\Delta\operatorname{vec}(\boldsymbol{S_R}')} \operatorname{span}(\boldsymbol{e_i})^{\perp}$, where $(\boldsymbol{e_i})_{i=1}^{rm}$ is the canonical basis of $\mathbb{C}^{rm}$.*

Proof. We have:

$$\begin{aligned}
&\operatorname{Ker}(\boldsymbol{I_m}\otimes\boldsymbol{X})\cap\left(\Sigma_{\operatorname{vec}(\boldsymbol{S_R})}-\Sigma_{\operatorname{vec}(\boldsymbol{S_R}')}\right)\subseteq\{0\}\\
\iff&\operatorname{Ker}(\boldsymbol{I_m}\otimes\boldsymbol{X})\cap\left(\overline{\Sigma}_{\operatorname{vec}(\boldsymbol{S_R})\cup\operatorname{vec}(\boldsymbol{S_R}')}\setminus\bigcup_{i\in\operatorname{vec}(\boldsymbol{S_R})\Delta\operatorname{vec}(\boldsymbol{S_R}')}\operatorname{span}(\boldsymbol{e_i})^{\perp}\right)\subseteq\{0\}\\
\iff&\operatorname{Ker}(\boldsymbol{I_m}\otimes\boldsymbol{X})\cap\overline{\Sigma}_{\operatorname{vec}(\boldsymbol{S_R})\cup\operatorname{vec}(\boldsymbol{S_R}')}\subseteq\{0\}\cup\bigcup_{i\in\operatorname{vec}(\boldsymbol{S_R})\Delta\operatorname{vec}(\boldsymbol{S_R}')}\operatorname{span}(\boldsymbol{e_i})^{\perp},
\end{aligned}\tag{A.1}$$

where the second line comes from the second assertion of Lemma 3.3, and the third line comes from the fact that for any sets A, B, C, we have the equivalence:

$$A\backslash B\subseteq C\iff A\subseteq B\cup C.\tag{A.2}$$

□

Remark. In the case where $\boldsymbol{S_R}=\boldsymbol{S_R'}$, the set of indices $\operatorname{vec}(\boldsymbol{S_R})\Delta\operatorname{vec}(\boldsymbol{S_R}')$ is empty, and condition 2 becomes:

$$\operatorname{Ker}(\boldsymbol{I_m}\otimes\boldsymbol{X})\cap\overline{\Sigma}_{\operatorname{vec}(\boldsymbol{S_R})}=\{0\},\tag{A.3}$$

which can be characterized using Lemma 3.5. In the case where $\boldsymbol{S_R}\neq\boldsymbol{S_R'}$, the set $\operatorname{vec}(\boldsymbol{S_R})\Delta\operatorname{vec}(\boldsymbol{S_R}')$ is not empty, and since $0\in\operatorname{span}(\boldsymbol{e_i})^{\perp}$ for all $i\in[\![rm]\!]$, condition 2 becomes:

$$\operatorname{Ker}(\boldsymbol{I_m}\otimes\boldsymbol{X})\cap\overline{\Sigma}_{\operatorname{vec}(\boldsymbol{S_R})\cup\operatorname{vec}(\boldsymbol{S_R}')}\subseteq\bigcup_{i\in\operatorname{vec}(\boldsymbol{S_R})\Delta\operatorname{vec}(\boldsymbol{S_R}')}\operatorname{span}(\boldsymbol{e_i})^{\perp}.\tag{A.4}$$

In the following, we will show that we can simplify condition (A.4).

Condition (A.4) is an inclusion of a subspace in a union of hyperplanes. In fact, Lemma A.3 below shows that this condition is equivalent to the inclusion of this subspace in one of these hyperplanes. Its proof relies on the following technical lemma.

Lemma A.2. *Denote $(\boldsymbol{e_i})_{i=1}^{p}$ the canonical basis of $\mathbb{C}^p$. Let $F\subseteq\mathbb{C}^p$ be a linear subspace, $I\subseteq[\![p]\!]$ be a subset of indices, $\boldsymbol{v}\in F\setminus\left(\bigcup_{i\in I}\operatorname{span}(\boldsymbol{e_i})^{\perp}\right)$ be a vector, and $k\in[\![p]\!]\backslash I$ an index. Consider $\boldsymbol{y}\in F\setminus\left(\operatorname{span}(\boldsymbol{e_k})^{\perp}\right)$. Then, there exists $\lambda\in\mathbb{C}$ such that:*

$$\boldsymbol{v}+\lambda\boldsymbol{y}\in F\setminus\left(\bigcup_{i\in I\cup\{k\}}\operatorname{span}(\boldsymbol{e_i})^{\perp}\right).\tag{A.5}$$

Proof. Let $\lambda\in\mathbb{C}\backslash\{-\frac{\boldsymbol{v}_i}{\boldsymbol{y}_i}\mid i\in(I\cup\{k\})\cap\operatorname{supp}(\boldsymbol{y})\}$. Then, we have:

$$(\boldsymbol{v}+\lambda\boldsymbol{y})_k=\boldsymbol{v}_k+\lambda\boldsymbol{y_k}\neq0,\tag{A.6}$$

since $\lambda\neq-\frac{\boldsymbol{v}_k}{\boldsymbol{y}_k}$. And, for all $i\in I$, we have:

$$(\boldsymbol{v}+\lambda\boldsymbol{y})_i=\boldsymbol{v}_i+\lambda\boldsymbol{y_i}=\begin{cases}\boldsymbol{v}_i & \text{if } \boldsymbol{y}_i=0\\ \boldsymbol{v}_i+\lambda\boldsymbol{y}_i & \text{otherwise}\end{cases}.\tag{A.7}$$

If $\boldsymbol{y}_i=0$, then we have $\boldsymbol{v}_i\neq0$ since $i\in\operatorname{supp}(\boldsymbol{v})$. And if $\boldsymbol{y}_i=0$, then we have $\boldsymbol{v}_i+\lambda\boldsymbol{y}_i\neq0$ since $\lambda\neq-\frac{\boldsymbol{v}_i}{\boldsymbol{y}_i}$. This gives $(\boldsymbol{v}+\lambda\boldsymbol{y})_i\neq0$. Therefore, $\boldsymbol{v}+\lambda\boldsymbol{y}\notin\bigcup_{i\in I\cup\{k\}}\operatorname{span}(\boldsymbol{e_i})^{\perp}$, and since $\boldsymbol{v},\boldsymbol{y}\in F$, we have $\boldsymbol{v}+\lambda\boldsymbol{y}\in F\setminus\left(\bigcup_{i\in I\cup\{k\}}\operatorname{span}(\boldsymbol{e_i})^{\perp}\right)$. □

Now we can show the simplification of condition (A.4).

Lemma A.3. (⋆) *Let $\boldsymbol{S_R},\boldsymbol{S_R}'\in\mathbb{B}^{r\times m}$ be two right supports, and $\boldsymbol{X}\in\mathbb{C}^{n\times r}$ a fixed left factor. Suppose that $\boldsymbol{S_R}\neq\boldsymbol{S_R'}$. Then, the following assertions are equivalent:*

1. $\operatorname{Ker}(\boldsymbol{I_m}\otimes\boldsymbol{X})\cap\overline{\Sigma}_{\operatorname{vec}(\boldsymbol{S_R})\cup\operatorname{vec}(\boldsymbol{S_R}')} \subseteq \bigcup_{i\in\operatorname{vec}(\boldsymbol{S_R})\Delta\operatorname{vec}(\boldsymbol{S_R}')}\operatorname{span}(\boldsymbol{e_i})^{\perp}$, *where* $(\boldsymbol{e_i})_{i=1}^{rm}$ *is the canonical basis of* $\mathbb{C}^{rm}$*;*

2. *there exists an index* $i\in\operatorname{vec}(\boldsymbol{S_R})\Delta\operatorname{vec}(\boldsymbol{S_R}')$ *such that:*

$$\operatorname{Ker}(\boldsymbol{I_m}\otimes\boldsymbol{X})\cap\overline{\Sigma}_{\operatorname{vec}(\boldsymbol{S_R})\cup\operatorname{vec}(\boldsymbol{S_R}')} \subseteq \operatorname{span}(\boldsymbol{e_i})^{\perp}. \tag{A.8}$$

Proof. The proof of the contraposition (non 2 $\Rightarrow$ non 1) is essentially an iterative contruction of a vector $\boldsymbol{v}\in\operatorname{Ker}(\boldsymbol{I_m}\otimes\boldsymbol{X})\cap\overline{\Sigma}_{\operatorname{vec}(\boldsymbol{S_R})\cup\operatorname{vec}(\boldsymbol{S_R}')}$ with nonzero entries for the indices in the subset $\operatorname{vec}(\boldsymbol{S_R})\Delta\operatorname{vec}(\boldsymbol{S_R}')$, from the assumption (non 2). At each iteration, the construction uses Lemma A.2. The formal proof is deferred to the next chapter. $\square$

In fact, there is a more intuitive way to interpret the inclusion (A.8), in terms of columns of the matrix $(\boldsymbol{I_m}\otimes\boldsymbol{X})$. Indeed, according to the following lemma, this inclusion is equivalent to the linear independence of the i-th column of $(\boldsymbol{I_m}\otimes\boldsymbol{X})$ from the columns indexed by $\operatorname{vec}(\boldsymbol{S_R})\cup\operatorname{vec}(\boldsymbol{S_R}')\setminus\{i\}$.

Lemma A.4. *Let* $\boldsymbol{M}\in\mathbb{C}^{p\times q}$ *be a matrix,* $T\subseteq[\![q]\!]$ *be a subset of column indices, and* $l\in T$ *be a column index. Then, we have the following equivalence:*

$$\operatorname{Ker}(\boldsymbol{M})\cap\overline{\Sigma}_T\subseteq\operatorname{span}(\boldsymbol{e_l})^{\perp} \iff \boldsymbol{M}_{\bullet l}\notin\operatorname{span}(\boldsymbol{M}_{\bullet j})_{j\in T\setminus\{l\}}. \tag{A.9}$$

In other words, we have the inclusion $\operatorname{Ker}(\boldsymbol{M})\cap\overline{\Sigma}_T\subseteq\operatorname{span}(\boldsymbol{e_l})^{\perp}$ *if, and only if, the* l*-th column of* $\boldsymbol{M}$ *is linearly independent from the columns* $\left\{\boldsymbol{M}_{\bullet j}\mid j\in T\setminus\{l\}\right\}$*.*

Proof. We will show the contraposition of (A.9). Suppose $\operatorname{Ker}(\boldsymbol{M})\cap\overline{\Sigma}_T\not\subseteq\operatorname{span}(\boldsymbol{e_l})^{\perp}$. Then, there exists $\boldsymbol{y}\in\operatorname{Ker}(\boldsymbol{M})\cap\overline{\Sigma}_T$ such that $\boldsymbol{y}_l\neq 0$. Since:

$$0=\boldsymbol{M}\boldsymbol{y}=\boldsymbol{y}_l\boldsymbol{M}_{\bullet l}+\sum_{j\in T\setminus\{l\}}\boldsymbol{y}_j\boldsymbol{M}_{\bullet j}, \tag{A.10}$$

we have $\boldsymbol{M}_{\bullet l}=-\sum_{j\in T\setminus\{l\}}\frac{\boldsymbol{y}_j}{\boldsymbol{y}_l}\boldsymbol{M}_{\bullet j}\in\operatorname{span}(\boldsymbol{M}_{\bullet j})_{j\in T\setminus\{l\}}$.

Conversely, suppose that $\boldsymbol{M}_{\bullet l}\in\operatorname{span}(\boldsymbol{M}_{\bullet j})_{j\in T\setminus\{l\}}$. Then, we fix $(\lambda_j)_{j\in T\setminus\{l\}}\in\mathbb{C}^{T\setminus\{l\}}$ such that $\boldsymbol{M}_{\bullet l}=\sum_{j\in T\setminus\{l\}}\lambda_j\boldsymbol{M}_{\bullet j}$. Then, define $\boldsymbol{y}\in\overline{\Sigma}_T$ such that:

$$\forall j\in T,\quad \boldsymbol{y}_j=\begin{cases}\lambda_j & \text{if } j\neq l\\ -1 & \text{otherwise}\end{cases}. \tag{A.11}$$

By construction, we have $\boldsymbol{y}\in\operatorname{Ker}(\boldsymbol{M})$, which means that $\boldsymbol{y}\in\operatorname{Ker}(\boldsymbol{M})\cap\overline{\Sigma}_T$. However, $\boldsymbol{y}_l=-1$, so $\boldsymbol{y}\notin\operatorname{span}(\boldsymbol{e_l})^{\perp}$. This means that $\operatorname{Ker}(\boldsymbol{M})\cap\overline{\Sigma}_T\not\subseteq\operatorname{span}(\boldsymbol{e_l})^{\perp}$. $\square$

Because of the block structure of the matrix $(\boldsymbol{I_m}\otimes\boldsymbol{X})$, the linear independence of a column in $(\boldsymbol{I_m}\otimes\boldsymbol{X})$ from other columns in $(\boldsymbol{I_m}\otimes\boldsymbol{X})$ can be reduced to the linear independence of a column in $\boldsymbol{X}$ from other columns in $\boldsymbol{X}$. This is precisely the claim of the following lemma.

Lemma A.5. ($\star$) *Let* $\boldsymbol{S_R},\boldsymbol{S_R}'\in\mathbb{B}^{r\times m}$ *be two right supports, and* $\boldsymbol{X}\in\mathbb{C}^{n\times r}$ *a fixed left factor. Suppose that* $\boldsymbol{S_R}\neq\boldsymbol{S}'_{\boldsymbol{R}}$*. Then, the following assertions are equivalent:*

1. *there exists an index* $i\in\operatorname{vec}(\boldsymbol{S_R})\Delta\operatorname{vec}(\boldsymbol{S_R}')$ *such that the* i*-th column of* $(\boldsymbol{I_m}\otimes\boldsymbol{X})$ *is linearly independent from the columns* $\left\{(\boldsymbol{I_m}\otimes\boldsymbol{X})_{\bullet i'}\mid i'\in(\operatorname{vec}(\boldsymbol{S_R})\cup\operatorname{vec}(\boldsymbol{S_R}))\setminus\{i\}\right\}$*;*

2. *there exists a column index* $l\in[\![m]\!]$ *and a row index* $k\in(\boldsymbol{S_R})_{\bullet l}\Delta(\boldsymbol{S_R}')_{\bullet l}$ *such that the* k*-th column of* $\boldsymbol{X}$ *is linearly independent from the columns* $\left\{\boldsymbol{X}_{\bullet j}\mid j\in((\boldsymbol{S_R})_{\bullet l}\cup(\boldsymbol{S}'_{\boldsymbol{R}})_{\bullet l})\setminus\{k\}\right\}$*.*

Proof. The proof is essentially a succession of reformulations, so we defer it to the next chapter. $\square$

In conclusion, from the previous lemmas, we obtain the following theorem characterizing global exact right identifiability (Definition 3.2), for a general family of allowed right supports.

Theorem A.1. *Let $\Omega_R \subseteq \mathbb{B}^{r\times m}$ be a family of allowed right supports, and $\boldsymbol{X} \in \mathbb{C}^{n\times r}$ a fixed left factor. Then, $(\Omega_R, \boldsymbol{X})$ is globally and exactly right identifiable (Definition 3.2) if, and only if, both of the following conditions are verified:*

1. *for all $\boldsymbol{S_R} \in \Omega_R$, for all column indices $l \in [\![m]\!]$, the columns $\{\boldsymbol{X}_{\bullet j} \mid j \in (\boldsymbol{S_R})_{\bullet l}\}$ are linearly independent.*
2. *for all $\boldsymbol{S_R}, \boldsymbol{S_R}' \in \Omega_R$ where $\boldsymbol{S_R} \neq \boldsymbol{S_R}'$, there exists a column index $l \in [\![m]\!]$ and a row index $k \in (\boldsymbol{S_R})_{\bullet l} \Delta (\boldsymbol{S_R}')_{\bullet l}$ such that the k-th column of $\boldsymbol{X}$ is linearly independent from the columns $\left\{\boldsymbol{X}_{\bullet j} \mid j \in ((\boldsymbol{S_R})_{\bullet l} \cup (\boldsymbol{S}'_{\boldsymbol{R}})_{\bullet l}) \setminus \{k\}\right\}$.*

Proof. By Proposition A.1, $(\Omega_R, \boldsymbol{X})$ is globally and exactly right identifiable (Definition 3.2) if, and only if,

$$\forall \boldsymbol{S_R}, \boldsymbol{S_R}' \in \Omega_R,\ \mathrm{Ker}(\boldsymbol{I_m} \otimes \boldsymbol{X}) \cap \left(\Sigma_{\mathrm{vec}(\boldsymbol{S_R})} - \Sigma_{\mathrm{vec}(\boldsymbol{S_R}')}\right) \subseteq \{0\} \tag{A.12}$$

which is equivalent to

$$\begin{cases} \forall \boldsymbol{S_R} \in \Omega_R,\ \mathrm{Ker}(\boldsymbol{I_m} \otimes \boldsymbol{X}) \cap \left(\Sigma_{\mathrm{vec}(\boldsymbol{S_R})} - \Sigma_{\mathrm{vec}(\boldsymbol{S_R})}\right) \subseteq \{0\} \\ \forall \boldsymbol{S_R}, \boldsymbol{S_R}' \in \Omega_R \text{ s.t. } \boldsymbol{S_R} \neq \boldsymbol{S_R}',\ \mathrm{Ker}(\boldsymbol{I_m} \otimes \boldsymbol{X}) \cap \left(\Sigma_{\mathrm{vec}(\boldsymbol{S_R})} - \Sigma_{\mathrm{vec}(\boldsymbol{S_R})}\right) \subseteq \{0\} \end{cases} \tag{A.13}$$

Then, we have, for all $\boldsymbol{S_R} \in \Omega_R$, we have:

$$\begin{aligned} & \mathrm{Ker}(\boldsymbol{I_m} \otimes \boldsymbol{X}) \cap \left(\Sigma_{\mathrm{vec}(\boldsymbol{S_R})} - \Sigma_{\mathrm{vec}(\boldsymbol{S_R})}\right) \subseteq \{0\} \\ \iff & \mathrm{Ker}(\boldsymbol{I_m} \otimes \boldsymbol{X}) \cap \overline{\Sigma}_{\mathrm{vec}(\boldsymbol{S_R})} \subseteq \{0\} \text{ by Lemma 3.3} \\ \iff & \mathrm{Ker}(\boldsymbol{I_m} \otimes \boldsymbol{X}) \cap \overline{\Sigma}_{\mathrm{vec}(\boldsymbol{S_R})} = \{0\} \text{ because } \overline{\Sigma}_{\mathrm{vec}(\boldsymbol{S_R})} \text{ is a linear space} \\ \iff & \forall l \in [\![m]\!],\ \mathrm{Ker}(\boldsymbol{X}) \cap \overline{\Sigma}_{(\boldsymbol{S_R})_{\bullet l}} = \{0\} \text{ by Lemma 3.5} \\ \iff & \text{Condition 1.} \end{aligned} \tag{A.14}$$

And, for all $\boldsymbol{S_R}, \boldsymbol{S_R}' \in \Omega_R$ such that $\boldsymbol{S_R} \neq \boldsymbol{S_R}'$, we have:

$$\begin{aligned} & \mathrm{Ker}(\boldsymbol{I_m} \otimes \boldsymbol{X}) \cap \left(\Sigma_{\mathrm{vec}(\boldsymbol{S_R})} - \Sigma_{\mathrm{vec}(\boldsymbol{S_R})}\right) \subseteq \{0\} \\ \iff & \mathrm{Ker}(\boldsymbol{I_m} \otimes \boldsymbol{X}) \cap \overline{\Sigma}_{\mathrm{vec}(\boldsymbol{S_R}) \cup \mathrm{vec}(\boldsymbol{S_R}')} \subseteq \bigcup_{i \in \mathrm{vec}(\boldsymbol{S_R}) \Delta\, \mathrm{vec}(\boldsymbol{S_R}')} \mathrm{span}(\boldsymbol{e_i})^{\perp} \text{ by Lemma A.1} \\ \iff & \exists i \in \mathrm{vec}(\boldsymbol{S_R}) \Delta\, \mathrm{vec}(\boldsymbol{S}'_{\boldsymbol{R}}),\ \mathrm{Ker}(\boldsymbol{I_m} \otimes \boldsymbol{X}) \cap \overline{\Sigma}_{\mathrm{vec}(\boldsymbol{S_R}) \cup \mathrm{vec}(\boldsymbol{S_R}')} \subseteq \mathrm{span}(\boldsymbol{e_i})^{\perp} \text{ by Lemma A.3} \\ \iff & \exists i \in \mathrm{vec}(\boldsymbol{S_R}) \Delta\, \mathrm{vec}(\boldsymbol{S}'_{\boldsymbol{R}}),\ (\boldsymbol{I_m} \otimes \boldsymbol{X})_{\bullet i} \notin \mathrm{span}\left\{(\boldsymbol{I_m} \otimes \boldsymbol{X})_{\bullet i'}\right\}_{i' \in (\mathrm{vec}(\boldsymbol{S_R}) \cup \mathrm{vec}(\boldsymbol{S_R})) \setminus \{i\}} \text{ by Lemma A.4} \\ \iff & \text{Condition 2 by Lemma A.5.} \end{aligned} \tag{A.15}$$

Combining all these equivalences, we obtain the proof of the theorem. □

A.1.2 Application to closed families of right supports

Theorem A.1 can be applied to the specific case where the family of right supports is closed, *i.e.*, $\Omega_R = \overline{\Omega}_R$. In this case, it is possible to reduce directly condition 1 and 2 of Theorem A.1 to condition 5 of Theorem A.1, by using the following lemma.

Lemma A.6. ($\star$) *Let $\Omega_R \subseteq \mathbb{B}^{r\times m}$ be a closed family of allowed right supports, in the sense that $\Omega_R = \overline{\Omega}_R$, and $\boldsymbol{X} \in \mathbb{C}^{n\times r}$ be a fixed left factor. Then, the following assertions are equivalent:*

1. *for all* $\boldsymbol{S_R} \in \Omega_R$*, the columns* $\left\{(\boldsymbol{I_m} \otimes \boldsymbol{X})_{\bullet i'} \mid i' \in \operatorname{vec}(\boldsymbol{S_R})\right\}$ *are linearly independent, and for all* $\boldsymbol{S_R}, \boldsymbol{S'_R} \in \Omega_R$ *such that* $\boldsymbol{S_R} \neq \boldsymbol{S'_R}$*, there exists an index* $i \in \operatorname{vec}(\boldsymbol{S_R}) \Delta \operatorname{vec}(\boldsymbol{S_R}')$ *such that the* i*-th column of* $(\boldsymbol{I_m} \otimes \boldsymbol{X})$ *is linearly independent from the columns* $\left\{(\boldsymbol{I_m} \otimes \boldsymbol{X})_{\bullet i'} \mid i' \in \operatorname{vec}(\boldsymbol{S_R}) \cup \operatorname{vec}(\boldsymbol{S_R}) \backslash \{i\}\right\}$*;*

2. *for all* $\boldsymbol{S_R}, \boldsymbol{S'_R} \in \Omega_R$*, the columns* $\left\{(\boldsymbol{I_m} \otimes \boldsymbol{X})_{\bullet i'} \mid i' \in \operatorname{vec}(\boldsymbol{S_R}) \cup \operatorname{vec}(\boldsymbol{S'_R})\right\}$ *are linearly independent.*

Proof sketch. The spirit is the following one. Consider $\boldsymbol{S_R}, \boldsymbol{S'_R} \in \Omega_R$ two allowed right supports, such that $\boldsymbol{S_R} \neq \boldsymbol{S'_R}$. Denote $T := \operatorname{vec}(\boldsymbol{S_R}) \cup \operatorname{vec}(\boldsymbol{S'_R})$ and $D := \operatorname{vec}(\boldsymbol{S_R}) \Delta \operatorname{vec}(\boldsymbol{S'_R})$. The idea is to construct iteratively from the second part of condition 1 some indices $\{i_1, ..., i_{\operatorname{card}(D)}\} \subseteq D$ to show that the columns of $(\boldsymbol{I_m} \otimes \boldsymbol{X})$ indexed by D are linearly independent, and the i-th column of $(\boldsymbol{I_m} \otimes \boldsymbol{X})$ is linearly independent from the columns indexed by $T \backslash D$ for all $i \in D$. This is possible because of the assumption $\Omega_R = \overline{\Omega}_R$. Then, by applying the first part of condition 1, we show that the columns indexed by $T \backslash D$ are linearly independent, which show that all the columns of $(\boldsymbol{I_m} \otimes \boldsymbol{X})$ indexed by T are linearly independent. As the formal proof is slightly long, we defer it to the next chapter. □

Then, the following corollary is the application of Theorem A.1 to obtain the claim of Theorem 3.1 in the specific case where the considered family of right supports is closed.

Corollary A.1. *Let* $\Omega_R \subseteq \mathbb{B}^{r \times m}$ *be a closed family of allowed right supports, in the sense that* $\Omega_R = \overline{\Omega}_R$*, and* $\boldsymbol{X} \in \mathbb{C}^{n \times r}$ *a fixed left factor. Then, the two conditions of Theorem 3.1 are verified if, and only if, condition 5 of Theorem 3.1 is verified.*

Proof. Suppose $\Omega_R = \overline{\Omega}_R$. Then we have the following equivalences:

$$
\begin{aligned}
&\text{Condition 1 and 2 of Theorem A.1} \\
\iff &\begin{cases} \forall \boldsymbol{S_R} \in \Omega_R,\ \forall l \in [\![m]\!],\ \operatorname{Ker}(\boldsymbol{X}) \cap \overline{\Sigma}_{(\boldsymbol{S_R})_{\bullet l}} = \{0\} \\ \forall \boldsymbol{S_R}, \boldsymbol{S'_R} \in \Omega_R \text{ s.t. } \boldsymbol{S_R} \neq \boldsymbol{S'_R},\ \exists i \in \operatorname{vec}(\boldsymbol{S_R}) \Delta \operatorname{vec}(\boldsymbol{S'_R}), \\ \qquad (\boldsymbol{I_m} \otimes \boldsymbol{X})_{\bullet i} \notin \operatorname{span}\left\{(\boldsymbol{I_m} \otimes \boldsymbol{X})_{\bullet i'} \mid i' \in \operatorname{vec}(\boldsymbol{S_R}) \cup \operatorname{vec}(\boldsymbol{S_R}) \backslash \{i\}\right\} \end{cases} && \text{by Lemma A.5} \\
\iff &\begin{cases} \forall \boldsymbol{S_R} \in \Omega_R,\ \operatorname{Ker}(\boldsymbol{I_m} \otimes \boldsymbol{X}) \cap \overline{\Sigma}_{\operatorname{vec}(\boldsymbol{S_R}) \cup \operatorname{vec}(\boldsymbol{S'_R})} = \{0\} \\ \forall \boldsymbol{S_R}, \boldsymbol{S'_R} \in \Omega_R \text{ s.t. } \boldsymbol{S_R} \neq \boldsymbol{S'_R},\ \exists i \in \operatorname{vec}(\boldsymbol{S_R}) \Delta \operatorname{vec}(\boldsymbol{S'_R}), \\ \qquad (\boldsymbol{I_m} \otimes \boldsymbol{X})_{\bullet i} \notin \operatorname{span}\left\{(\boldsymbol{I_m} \otimes \boldsymbol{X})_{\bullet i'} \mid i' \in \operatorname{vec}(\boldsymbol{S_R}) \cup \operatorname{vec}(\boldsymbol{S_R}) \backslash \{i\}\right\} \end{cases} && \text{by Lemma 3.5} \\
\iff &\forall \boldsymbol{S_R}, \boldsymbol{S'_R} \in \Omega_R,\ \operatorname{Ker}(\boldsymbol{I_m} \otimes \boldsymbol{X}) \cap \overline{\Sigma}_{\operatorname{vec}(\boldsymbol{S_R}) \cup \operatorname{vec}(\boldsymbol{S'_R})} = \{0\} \text{ by Lemma A.6} \\
\iff &\forall \boldsymbol{S_R}, \boldsymbol{S'_R} \in \Omega_R,\ \forall l \in [\![m]\!],\ \operatorname{Ker}(\boldsymbol{X}) \cap \overline{\Sigma}_{(\boldsymbol{S_R})_{\bullet l} \cup (\boldsymbol{S'_R})_{\bullet l}} = \{0\} \text{ by Lemma 3.5} \\
\iff &\text{Condition 5 of Theorem 3.1.}
\end{aligned}
\tag{A.16}
$$

□

A.2 Expression of the restricted rank 2 null space

This extension was proposed in Section 4.4.3 in the main text. One natural question is to understand whether or not it is possible to simplify the expression of the restricted rank 2 null space $\mathcal{N}(\mathscr{S}) \cap \prod_{i=1}^{r} \Delta_{\boldsymbol{S_i},1}$. In particular, we would like to simplify the expression of the secant sets $\Delta_{\boldsymbol{S_i},1}$ for $i \in [\![r]\!]$. To answer this question, we first introduce Lemma A.7 which shows a simpler formulation for the extended secant set $\overline{\Delta}_{\boldsymbol{S_i},1}$.

Lemma A.7. *Let* $\boldsymbol{S} \in \mathbb{B}^{n \times m} \cap \mathcal{R}_1$ *be a rank 1 support. Then, we have* $\overline{\Delta}_{\boldsymbol{S},1} = \overline{\Sigma}_{\boldsymbol{S},2}$*.*

Proof. Let $\boldsymbol{X} \in \overline{\Delta}_{\boldsymbol{S},1}$. Then, there exists $\boldsymbol{C}, \boldsymbol{D} \in \overline{\Sigma}_{\boldsymbol{S},1}$ such that $\boldsymbol{X} = \boldsymbol{C} - \boldsymbol{D}$. But $\operatorname{rank}(\boldsymbol{X}) \leq \operatorname{rank}(\boldsymbol{C}) + \operatorname{rank}(\boldsymbol{D}) \leq 2$, and $\operatorname{supp}(\boldsymbol{X}) = \operatorname{supp}(\boldsymbol{C} - \boldsymbol{D}) \subseteq \operatorname{supp}(\boldsymbol{C}) \cup \operatorname{supp}(\boldsymbol{D}) = \boldsymbol{S}$, which means that $\boldsymbol{X} \in \overline{\Sigma}_{\boldsymbol{S},2}$. Conversely, let $\boldsymbol{X} \in \overline{\Sigma}_{\boldsymbol{S},2}$. Since $\boldsymbol{S}$ is a rank 1 support, the matrix $\boldsymbol{X}$ restricted on $\boldsymbol{S}$ denoted $\boldsymbol{X}_{|\boldsymbol{S}}$ is actually a submatrix of $\boldsymbol{X}$. Because $\boldsymbol{X}$ is at most of rank 2 and $\operatorname{supp}(\boldsymbol{X}) \subseteq \boldsymbol{S}$, $\boldsymbol{X}_{|\boldsymbol{S}}$ is also at most of rank 2. Then, by applying singular value decomposition on the submatrix $\boldsymbol{X}_{|\boldsymbol{S}}$, we obtain $\boldsymbol{X} \in \overline{\Delta}_{\boldsymbol{S},1}$. □

Can we hope to obtain such simplification for the non-extended secant set $\Delta_{\boldsymbol{S},1}$? We can have the inclusion $\Delta_{\boldsymbol{S},1} \subseteq \overline{\Sigma}_{\boldsymbol{S},2}$ from Lemma A.8 below, but not the inverse inclusion as it is shown in Example A.1. Going back to the restricted rank 2 null space, we have the inclusion $\mathcal{N}(\mathscr{S}) \cap \prod_{i=1}^{r} \Delta_{\boldsymbol{S_i},1} \subseteq \mathcal{N}(\mathscr{S}, 2) \cap \prod_{i=1}^{r} \overline{\Sigma}_{\boldsymbol{S_i}}$ using Lemma A.8, but not the inverse inclusion.

Lemma A.8. *The following assertions hold.*

1. *Let $\boldsymbol{S} \in \mathbb{B}^{n\times m} \cap \mathcal{R}_1$ be a rank 1 support. Then, we have $\Delta_{\boldsymbol{S},1} \subsetneq \overline{\Sigma}_{\boldsymbol{S},2}$.*

2. *Let $\underline{\boldsymbol{S}} \in (\mathbb{B}^{n\times m} \cap \mathcal{R}_1)^r$ be a r-tuple of rank 1 supports. Then, we have $\mathcal{N}(\mathscr{S}) \cap \prod_{i=1}^{r} \Delta_{\boldsymbol{S_i},1} \subsetneq \mathcal{N}(\mathscr{S}, 2) \cap \prod_{i=1}^{r} \overline{\Sigma}_{\boldsymbol{S_i}}$.*

Proof. The proof of $\overline{\Delta}_{\boldsymbol{S},1} \subseteq \overline{\Sigma}_{\boldsymbol{S},2}$ is similar to the one in Lemma A.8. Let $\boldsymbol{X} \in \Delta_{\boldsymbol{S},1}$. Then, there exists $\boldsymbol{C}, \boldsymbol{D} \in \Sigma_{\boldsymbol{S},1}$ such that $\boldsymbol{X} = \boldsymbol{C} - \boldsymbol{D}$. But $\operatorname{rank}(\boldsymbol{X}) \leq \operatorname{rank}(\boldsymbol{C}) + \operatorname{rank}(\boldsymbol{D}) \leq 2$, and $\operatorname{supp}(\boldsymbol{X}) \subseteq \operatorname{supp}(\boldsymbol{C}) \cup \operatorname{supp}(\boldsymbol{D}) = S$, which means that $\boldsymbol{X} \in \overline{\Sigma}_{\boldsymbol{S},2}$. This shows that $\Delta_{\boldsymbol{S},1} \subseteq \overline{\Sigma}_{\boldsymbol{S},2}$, and Example A.1 shows that the inversion inclusion does not hold.

Now, let $\underline{\boldsymbol{X}} \in \mathcal{N}(\mathscr{S}) \cap \prod_{i=1}^{r} \Delta_{\boldsymbol{S_i},1}$. Then, for each $i \in [\![r]\!]$, we have $\boldsymbol{X_i} \in \Delta_{\boldsymbol{S_i},1}$, and by the first assertion, we obtain $\boldsymbol{X_i} \in \overline{\Sigma}_{\boldsymbol{S_i},2}$. This means that $\underline{\boldsymbol{X}} \in \mathcal{N}(\mathscr{S}, 2) \cap \prod_{i=1}^{r} \overline{\Sigma}_{\boldsymbol{S_i}}$, which shows that $\mathcal{N}(\mathscr{S}) \cap \prod_{i=1}^{r} \Delta_{\boldsymbol{S_i},1} \subseteq \mathcal{N}(\mathscr{S}, 2) \cap \prod_{i=1}^{r} \overline{\Sigma}_{\boldsymbol{S_i}}$. Example A.1 shows that the inversion inclusion does not hold. □

Example A.1 (Inverse inclusion in Lemma A.8 does not hold)**.** Define $\boldsymbol{X} := \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \end{pmatrix}$ and $\boldsymbol{S} := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Then, $\boldsymbol{X} \in \overline{\Sigma}_{\boldsymbol{S},2}$. Suppose that $\boldsymbol{X} \in \Delta_{\boldsymbol{S},1}$. Then, there exists $(\boldsymbol{a}, \boldsymbol{b}), (\boldsymbol{c}, \boldsymbol{b}) \in \mathbb{C}^2 \times \mathbb{C}^3$ such that $\boldsymbol{X} = \boldsymbol{a}\boldsymbol{b}^T - \boldsymbol{c}\boldsymbol{d}^T$, and for all $(i,j) \in [\![2]\!] \times [\![3]\!]$, we have $\boldsymbol{a}_i, \boldsymbol{c}_i \neq 0$ and $\boldsymbol{b}_j, \boldsymbol{d}_j \neq 0$. Now consider the matrix $\boldsymbol{M} = \begin{pmatrix} \boldsymbol{a}_1 & \boldsymbol{c}_1 \\ \boldsymbol{a}_2 & \boldsymbol{c}_2 \end{pmatrix}$. The rank of $\boldsymbol{M}$ is either 1 or 2, because $\boldsymbol{M} \neq 0$.

- Suppose $\operatorname{rank}(\boldsymbol{M}) = 2$. Then, taking the third column of the equality $\boldsymbol{X} = \boldsymbol{a}\boldsymbol{b}^T - \boldsymbol{c}\boldsymbol{d}^T$, we obtain:

$$\begin{cases} 0 = \boldsymbol{X}_{13} = \boldsymbol{a}_1\boldsymbol{b}_3 - \boldsymbol{c}_1\boldsymbol{d}_3 \\ 0 = \boldsymbol{X}_{23} = \boldsymbol{a}_2\boldsymbol{b}_3 - \boldsymbol{c}_2\boldsymbol{d}_3 \end{cases} \tag{A.17}$$

  which leads to the equality $\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \boldsymbol{M} \begin{pmatrix} \boldsymbol{b}_3 \\ -\boldsymbol{d}_3 \end{pmatrix}$. Since $\operatorname{rank}(\boldsymbol{M}) = 2$, we would have $\boldsymbol{b}_3 = \boldsymbol{d}_3 = 0$, which shows a contradiction.

- Now, suppose $\operatorname{rank}(\boldsymbol{M}) = 1$. Again, considering the first and second column of the equality $\boldsymbol{X} = \boldsymbol{a}\boldsymbol{b}^T - \boldsymbol{c}\boldsymbol{d}^T$, we have $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \boldsymbol{X}_{\bullet 1} = \boldsymbol{M} \begin{pmatrix} \boldsymbol{b_1} \\ -\boldsymbol{d}_1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \boldsymbol{X}_{\bullet 2} = \boldsymbol{M} \begin{pmatrix} \boldsymbol{b_2} \\ -\boldsymbol{d}_2 \end{pmatrix}$. Then, since $\operatorname{rank}(\boldsymbol{M}) = 1$, the vectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ would be colinear, which is also a contradiction.

Therefore, we conclude that $\boldsymbol{X} \notin \Delta_{\boldsymbol{S},1}$. This shows that $\overline{\Sigma}_{\boldsymbol{S},2} \neq \Delta_{\boldsymbol{S},1}$. Now, define $(\boldsymbol{S_1}, \boldsymbol{S_2}) := (\boldsymbol{S}, \boldsymbol{S})$. The previous paragraph showed that $\boldsymbol{X} \in \overline{\Sigma}_{\boldsymbol{S_1},2}$, but $\boldsymbol{X} \notin \Delta_{\boldsymbol{S_1},1}$. Similarly, we have $-\boldsymbol{X} \in \overline{\Sigma}_{\boldsymbol{S_2},2}$, but $-\boldsymbol{X} \notin \Delta_{\boldsymbol{S_2},1}$. Therefore, $(\boldsymbol{X}, -\boldsymbol{X}) \in \mathcal{N}(\mathscr{S}) \cap \prod_{i=1}^{2} \overline{\Sigma}_{\boldsymbol{S_i},2}$, but $(\boldsymbol{X}, -\boldsymbol{X}) \notin \mathcal{N}(\mathscr{S}) \cap \prod_{i=1}^{2} \Delta_{\boldsymbol{S_i},1}$. We conclude that $\mathcal{N}(\mathscr{S}) \cap \prod_{i=1}^{2} \overline{\Sigma}_{\boldsymbol{S_i},2} \neq \mathcal{N}(\mathscr{S}) \cap \prod_{i=1}^{2} \Delta_{\boldsymbol{S_i},1}$.

Non-pathological matrix structure

However, by ignoring some pathological matrix structure like the one shown in Example A.1, we can hope to have the equality $\Delta_{\boldsymbol{S},1} \cap \mathcal{K} = \overline{\Sigma}_{\boldsymbol{S},2} \cap \mathcal{K}$ where $\mathcal{K}$ is a set of matrices without pathological structure. In order to understand which set of non-pathological matrices $\mathcal{K}$ can be used, we first show some technical lemmas (Lemma A.9, Lemma A.10, Lemma A.11).

Lemma A.9. *Let $\boldsymbol{v}, \boldsymbol{u} \in \mathbb{C}^n$ be two vectors. Suppose there exists an index $i \in \operatorname{supp}(\boldsymbol{u}) \setminus \operatorname{supp}(\boldsymbol{v})$, i.e., $\boldsymbol{v}_i = 0$ but $\boldsymbol{u}_i \neq 0$. Then, there exists $\lambda \in \mathbb{C}$ such that $\operatorname{supp}(\boldsymbol{v}) \cup \{i\} \subseteq \operatorname{supp}(\boldsymbol{v} + \lambda \boldsymbol{u})$.*

Proof. Let $\lambda \in \mathbb{C} \setminus \left(\{0\} \cup \{-\frac{\boldsymbol{v}_k}{\boldsymbol{u}_k} \mid k \in \operatorname{supp}(\boldsymbol{v}) \cap \operatorname{supp}(\boldsymbol{u})\} \right)$. Then, we have $i \in \operatorname{supp}(\boldsymbol{v} + \lambda \boldsymbol{u})$, because $\boldsymbol{v}_i = 0$, $\boldsymbol{u}_i \neq 0$, $\lambda \neq 0$, so $\boldsymbol{v}_i + \lambda \boldsymbol{u}_i \neq 0$. And for $k \in \operatorname{supp}(\boldsymbol{v})$, we have $k \in \operatorname{supp}(\boldsymbol{v} + \lambda \boldsymbol{u})$, because:

- if $k \in \operatorname{supp}(\boldsymbol{v}) \setminus \operatorname{supp}(\boldsymbol{u})$, then we have $\boldsymbol{u}_k = 0$, $\boldsymbol{v}_k \neq 0$, so $\boldsymbol{v}_k + \lambda \boldsymbol{u}_k \neq 0$;
- and if $k \in \operatorname{supp}(\boldsymbol{v}) \cap \operatorname{supp}(\boldsymbol{u})$, then we have $\boldsymbol{v}_k \neq 0$, $\boldsymbol{u}_k \neq 0$, $\lambda \neq -\frac{\boldsymbol{v}_k}{\boldsymbol{u}_k}$, so $\boldsymbol{v}_k + \lambda \boldsymbol{u}_k \neq 0$.

This shows that $\operatorname{supp}(\boldsymbol{v}) \cup \{i\} \subseteq \operatorname{supp}(\boldsymbol{v} + \lambda \boldsymbol{u})$. □

Lemma A.10. *Let $\boldsymbol{X} \in \mathbb{C}^{n \times m}$ be a matrix. Suppose that $\boldsymbol{X}$ has no zero row. Then, for any $\boldsymbol{v} \in \operatorname{Im}(\boldsymbol{X})$ such that $\|\boldsymbol{v}\|_0 < n$, there exists a row index $i \in [\![n]\!] \setminus \operatorname{supp}(\boldsymbol{v})$, a column index $j \in [\![m]\!]$ and a scalar $\lambda \in \mathbb{C}$ such that $\|\boldsymbol{v}\|_0 + 1 \leq \|\boldsymbol{v} + \lambda \boldsymbol{X}_{\bullet j}\|_0$.*

Proof. Let $\boldsymbol{v} \in \operatorname{Im}(\boldsymbol{X})$ such that $\|\boldsymbol{v}\|_0 < n$. Then, $[\![n]\!] \setminus \operatorname{supp}(\boldsymbol{v})$ is non-empty, and there exists a row index $i \in [\![n]\!] \setminus \operatorname{supp}(\boldsymbol{v})$. Since the i-th row $\boldsymbol{X}_{i\bullet}$ is nonzero by assumption, there exists a column index $j \in [\![m]\!]$ such that $\boldsymbol{X}_{ij} \neq 0$. Then, $i \in \operatorname{supp}(\boldsymbol{X}_{\bullet j}) \setminus \operatorname{supp}(\boldsymbol{v})$, so by Lemma A.9, there exists $\lambda \in \mathbb{C}$ such that $\operatorname{supp}(\boldsymbol{v}) \cup \{i\} \subseteq \operatorname{supp}(\boldsymbol{v} + \lambda \boldsymbol{X}_{\bullet j})$. Since $i \notin \operatorname{supp}(\boldsymbol{v})$, we obtain $\|\boldsymbol{v}\|_0 + 1 = \operatorname{card}(\operatorname{supp}(\boldsymbol{v}) \cup \{i\}) \leq \|\boldsymbol{v} + \lambda \boldsymbol{X}_{\bullet j}\|_0$. □

Lemma A.11. ($\star$) *Let $\boldsymbol{X} \in \mathbb{C}^{n \times m}$ be a matrix. Suppose that $\boldsymbol{X}$ has no zero row. Then, we have:*

$$\operatorname{Im}(\boldsymbol{X}) \not\subseteq \bigcup_{i=1}^{n} \operatorname{span}(\boldsymbol{e_i})^{\perp} \tag{A.18}$$

where $(\boldsymbol{e_i})_{i=1}^n$ is the canonical basis of $\mathbb{C}^n$.

Proof sketch. Essentially, the proof of this result is an iterative construction of a vector $\boldsymbol{v}$ belonging to the image of $\boldsymbol{X}$ which does not have zero entries. This construction uses at each iteration Lemma A.10. As the formal proof is slightly long, we defer it to the next chapter. □

Then, based on the previous lemma, Proposition A.2 below claims that when a matrix $\boldsymbol{X}$ with a rank at most 2 has a structure such that $\boldsymbol{X}$ have no zero column and no zero row, we can decompose $\boldsymbol{X}$ as the difference $\boldsymbol{X} = \boldsymbol{C} - \boldsymbol{D}$, where $\boldsymbol{C}, \boldsymbol{D}$ are rank 1 matrices, with nonzero entries.

Proposition A.2. ($\star$) *Let $\boldsymbol{X} \in \mathbb{C}^{n \times m} \cap \mathcal{R}_2$ be a matrix with rank at most 2. Suppose that $\boldsymbol{X}$ has no zero column and no zero row. Then, there exists two rank 1 matrices $\boldsymbol{C}, \boldsymbol{D} \in \mathbb{C}^{n \times m} \cap \mathcal{R}_1$ such that $\operatorname{supp}(\boldsymbol{C}) = \operatorname{supp}(\boldsymbol{D}) = [\![n]\!] \times [\![m]\!]$, and $\boldsymbol{X} = \boldsymbol{C} - \boldsymbol{D}$.*

Proof sketch. The proof of this result is technical, so we defer it to the next chapter. But the spirit of the proof is the following: as we remark that the image $\operatorname{Im}(\boldsymbol{X})$ is at most of rank 2, we want to construct two vectors $\boldsymbol{a}, \boldsymbol{c}$ which:

- form a basis of a dimension 2 subspace containing the columns of $\boldsymbol{X}$;
- do not have zero entries;
- are not colinear to each column of $\boldsymbol{X}$.

This is possible thanks to the non-pathological structure considered (no zero column and no zero row in $\boldsymbol{X}$). Then, by expressing the columns of $\boldsymbol{X}$ in the basis $(\boldsymbol{a}, \boldsymbol{c})$, we show that $\boldsymbol{X}$ can be written as the difference $\boldsymbol{X} = \boldsymbol{C} - \boldsymbol{D}$ between two rank 1 matrices $\boldsymbol{C}, \boldsymbol{D}$, where $\mathrm{supp}(\boldsymbol{C}) = \mathrm{supp}(\boldsymbol{D}) = [\![n]\!] \times [\![m]\!]$. $\square$

Let $\boldsymbol{S} \in \mathbb{B}^{n\times m} \cap \mathcal{R}_1$ be a rank 1 support. We can then generalize Proposition A.2's result to find the set $\mathcal{K}$ such that $\Delta_{\boldsymbol{S},1} \cap \mathcal{K} = \overline{\Sigma}_{\boldsymbol{S},2} \cap \mathcal{K}$. Denote $\mathcal{K}_{\boldsymbol{S}}$ the set of matrices partially supported by $\boldsymbol{S}$, with zero column and no zero row in the rank 1 support $\boldsymbol{S}$:

$$\mathcal{K}_{\boldsymbol{S}} := \{\boldsymbol{M} \in \overline{\Sigma}_{\boldsymbol{S}} \mid \forall (i,j) \in \boldsymbol{S}, \boldsymbol{M}_{\bullet j} \neq 0, \boldsymbol{M}_{i\bullet} \neq 0\}. \tag{A.19}$$

Corollary A.2 uses $\mathcal{K}_{\boldsymbol{S}}$ as the set of non-pathological matrices, to obtain the equality $\Delta_{\boldsymbol{S},1} \cap \mathcal{K}_{\boldsymbol{S}} = \overline{\Sigma}_{\boldsymbol{S},2} \cap \mathcal{K}_{\boldsymbol{S}}$.

Corollary A.2. *The following assertions are verified.*

1. *Let $\boldsymbol{S} \in \mathbb{B}^{n\times m} \cap \mathcal{R}_1$ be a rank 1 support. Then, we have $\Delta_{\boldsymbol{S},1} \cap \mathcal{K}_{\boldsymbol{S}} = \overline{\Sigma}_{\boldsymbol{S},2} \cap \mathcal{K}_{\boldsymbol{S}}$.*

2. *Let $\underline{\boldsymbol{S}} \in (\mathbb{B}^{n\times m} \cap \mathcal{R}_1)^r$ be a r-tuple of rank 1 supports. Then, we have $\mathcal{N}(\mathscr{S}) \cap \prod_{i=1}^r (\Delta_{\boldsymbol{S}_i,1} \cap \mathcal{K}_{\boldsymbol{S}_i}) = \mathcal{N}(\mathscr{S}) \cap \prod_{i=1}^r (\overline{\Sigma}_{\boldsymbol{S}_i,2} \cap \mathcal{K}_{\boldsymbol{S}_i})$.*

Proof. Because of Lemma A.8, we only need to prove $\overline{\Sigma}_{\boldsymbol{S},2} \cap \mathcal{K}_{\boldsymbol{S}} \subseteq \Delta_{\boldsymbol{S},1} \cap \mathcal{K}_{\boldsymbol{S}}$. Let $\boldsymbol{X} \in \overline{\Sigma}_{\boldsymbol{S},2} \cap \mathcal{K}_{\boldsymbol{S}}$. Denote $(\boldsymbol{s_L}, \boldsymbol{s_R}) \in \mathbb{B}^n \times \mathbb{B}^m$ such that $\boldsymbol{S} = \boldsymbol{s_L}\boldsymbol{s_R}^T$. Since $\boldsymbol{S}$ is a rank 1 support, the matrix $\boldsymbol{X}$ restricted on $\boldsymbol{S}$ denoted $\boldsymbol{X}_{|\boldsymbol{S}} \in \mathbb{C}^{\boldsymbol{s_L}\times \boldsymbol{s_R}}$ is actually a submatrix of $\boldsymbol{X}$, and we can identify the space $\mathbb{C}^{\boldsymbol{s_L}\times \boldsymbol{s_R}}$ with $\mathbb{C}^{\|\boldsymbol{s_L}\|_0 \times \|\boldsymbol{s_R}\|_0}$. In particular, since $\boldsymbol{X} \in \mathcal{K}_{\boldsymbol{S}}$, there is no zero column and no zero row in the submatrix $\boldsymbol{X}_{|\boldsymbol{S}}$, and since $\boldsymbol{X}$ is at most of rank 2, $\boldsymbol{X}_{|\boldsymbol{S}}$ is also at most of rank 2. Therefore, we can apply Proposition A.2 to conclude that there exists some submatrices $\tilde{\boldsymbol{C}}, \tilde{\boldsymbol{D}} \in \mathbb{C}^{\boldsymbol{s_L}\times \boldsymbol{s_R}}$ such that $\boldsymbol{X}_{|\boldsymbol{S}} = \tilde{\boldsymbol{C}} - \tilde{\boldsymbol{D}}$ and all the entries of $\tilde{\boldsymbol{C}}, \tilde{\boldsymbol{D}}$ are nonzero. Then, by defining $\boldsymbol{C}, \boldsymbol{D} \in \Sigma_{\boldsymbol{S},1}$ as $\boldsymbol{C}_{|\boldsymbol{S}} = \tilde{\boldsymbol{C}}$ and $\boldsymbol{D}_{|\boldsymbol{S}} = \tilde{\boldsymbol{D}}$, we obtain $\boldsymbol{X} = \boldsymbol{C} - \boldsymbol{D}$, which shows that $\boldsymbol{X} \in \Delta_{\boldsymbol{S},1}$.

The second assertion is a direct consequence of the first assertion. $\square$

In fact, we observe that the complementary set $\overline{\Sigma}_{\boldsymbol{S},2} \backslash \mathcal{K}_{\boldsymbol{S}}$ is a null set for the Lebesgue measure in the linear space $\overline{\Sigma}_{\boldsymbol{S}}$, which means that the equality $\Delta_{\boldsymbol{S},1} = \overline{\Sigma}_{\boldsymbol{S},2}$ holds almost everywhere, in the sense of Corollary A.3.

Corollary A.3. *The following assertions hold.*

1. *Let $\boldsymbol{S} \in \mathbb{B}^{n\times m} \cap \mathcal{R}_1$ be a rank 1 support, and consider the Lebesgue measure on the linear space $\overline{\Sigma}_{\boldsymbol{S}}$. Then, for almost every $\boldsymbol{X} \in \overline{\Sigma}_{\boldsymbol{S}}$, we have $\boldsymbol{X} \in \Delta_{\boldsymbol{S},1} \iff \boldsymbol{X} \in \overline{\Sigma}_{\boldsymbol{S},2}$.*

2. *Let $\underline{\boldsymbol{S}} \in (\mathbb{B}^{n\times m} \cap \mathcal{R}_1)^r$ be a r-tuple of rank 1 supports, and consider now the Lebesgue measure on the linear space $\mathcal{N}(\mathscr{S}) \cap \prod_{i=1}^r \overline{\Sigma}_{\boldsymbol{S}_i}$. Then, for almost every $\underline{\boldsymbol{X}} \in \mathcal{N}(\mathscr{S}) \cap \prod_{i=1}^r \overline{\Sigma}_{\boldsymbol{S}_i}$, we have $\underline{\boldsymbol{X}} \in \mathcal{N}(\mathscr{S}) \cap \prod_{i=1}^r \Delta_{\boldsymbol{S}_i,1} \iff \underline{\boldsymbol{X}} \in \mathcal{N}(\mathscr{S}, 2) \cap \prod_{i=1}^r \overline{\Sigma}_{\boldsymbol{S}_i}$.*

Proof. To show the first assertion, let $\boldsymbol{X} \in \overline{\Sigma}_{\boldsymbol{S}}$ such that "$\boldsymbol{X} \in \Delta_{\boldsymbol{S},1} \iff \boldsymbol{X} \in \overline{\Sigma}_{\boldsymbol{S},2}$" is not verified. This is equivalent to say that the implication "$\boldsymbol{X} \in \overline{\Sigma}_{\boldsymbol{S},2} \Rightarrow \boldsymbol{X} \in \Delta_{\boldsymbol{S},1}$" is not verified, because by Lemma A.8, the inclusion $\Delta_{\boldsymbol{S},1} \subseteq \overline{\Sigma}_{\boldsymbol{S},2}$ holds. In other words, we have $\boldsymbol{X} \in \overline{\Sigma}_{\boldsymbol{S},2}$ and $\boldsymbol{X} \notin \Delta_{\boldsymbol{S},1}$. Suppose that $\boldsymbol{X} \in \mathcal{K}_{\boldsymbol{S}}$. Then, $\boldsymbol{X} \in \overline{\Sigma}_{\boldsymbol{S},2} \cap \mathcal{K}_{\boldsymbol{S}}$. But by Corollary A.2, we have $\overline{\Sigma}_{\boldsymbol{S},2} \cap \mathcal{K}_{\boldsymbol{S}} = \Delta_{\boldsymbol{S},1} \cap \mathcal{K}_{\boldsymbol{S}}$, which would mean that $\boldsymbol{X} \in \Delta_{\boldsymbol{S},1} \cap \mathcal{K}_{\boldsymbol{S}}$. This is a contradiction with the fact that $\boldsymbol{X} \notin \Delta_{\boldsymbol{S},1}$, so we conclude that $\boldsymbol{X} \in \overline{\Sigma}_{\boldsymbol{S},2} \backslash \mathcal{K}_{\boldsymbol{S}}$, and more generally, $\boldsymbol{X} \in \overline{\Sigma}_{\boldsymbol{S}} \backslash \mathcal{K}_{\boldsymbol{S}}$ because of the inclusion $\overline{\Sigma}_{\boldsymbol{S},2} \subseteq \overline{\Sigma}_{\boldsymbol{S}}$. However, we remark that $\overline{\Sigma}_{\boldsymbol{S}} \backslash \mathcal{K}_{\boldsymbol{S}}$ is a union of a finite number of linear subspaces with a dimension lesser than the dimension of $\overline{\Sigma}_{\boldsymbol{S}}$. Indeed, each of these subspaces corresponds to a subspace of matrices containing at least a zero column or a zero row, and therefore has a dimension lesser than the one of $\overline{\Sigma}_{\boldsymbol{S}}$. Then, the Lebesgue measure of such linear subspaces is zero. We conclude that $\overline{\Sigma}_{\boldsymbol{S}} \backslash \mathcal{K}_{\boldsymbol{S}}$ is a null set for the Lebesgue measure in the linear space $\overline{\Sigma}_{\boldsymbol{S}}$, which ends the proof. We use a similar reasoning to prove the second assertion. $\square$

Appendix B

Proofs

B.1 Proofs for Section 2.3 (Analyzing the notion of identifiability)

Proposition 2.1. *Let $\hat{\Omega} \subseteq \mathbb{B}^{n\times r} \times \mathbb{B}^{r\times m}$ be a family of allowed pairs of supports stable by permutation (Definition 2.2), and $(\boldsymbol{X}, \boldsymbol{Y}) \in \Sigma_{\hat{\Omega}}$ be a pair of factors. Then, $(\boldsymbol{X}, \boldsymbol{Y})$ is identifiable for the family $\hat{\Omega}$ (Definition 2.3) if, and only if, the following conditions are verified:*

1. *for all $(\boldsymbol{A}, \boldsymbol{B}) \in \Sigma_{\hat{\Omega}}$ verifying $\boldsymbol{AB} = \boldsymbol{XY}$, there exists a scaled permutation matrix $\boldsymbol{C} \in \mathbb{C}^{r\times r}$ (Definition 2.1) such that $\boldsymbol{A} = \boldsymbol{XC}$;*

2. *$\boldsymbol{Y}$ is right identifiable for the family $\Omega_R(\boldsymbol{X})$ and the left factor $\boldsymbol{X}$ (Definition 2.5).*

Proof. For proving necessity, suppose that $(\boldsymbol{X}, \boldsymbol{Y})$ is identifiable for $\hat{\Omega}$. Then, by definition of identifiability, for any $(\boldsymbol{A}, \boldsymbol{B}) \in \Sigma_{\hat{\Omega}}$ verifying $\boldsymbol{AB} = \boldsymbol{XY}$, we have $(\boldsymbol{A}, \boldsymbol{B}) \sim (\boldsymbol{X}, \boldsymbol{Y})$, which precisely means that there exists a scaled permutation matrix $\boldsymbol{C} \in \mathbb{C}^{r\times r}$ such that $\boldsymbol{A} = \boldsymbol{XC}$ and $\boldsymbol{B} = \boldsymbol{C}^{-1}\boldsymbol{Y}$. This shows condition 1. Then, condition 2 is verified by Lemma 2.1.

For proving sufficiency, suppose that condition 1 and 2 are verified, and let $(\boldsymbol{A}, \boldsymbol{B}) \in \Sigma_{\hat{\Omega}}$ verifying $\boldsymbol{AB} = \boldsymbol{XY}$. Then, by condition 1, there exists a scaled permutation matrix $\boldsymbol{C} \in \mathbb{C}^{r\times r}$ such that $\boldsymbol{A} = \boldsymbol{XC}$; or equivalently, $\boldsymbol{AC}^{-1} = \boldsymbol{X}$. We define $\boldsymbol{D} \in \mathbb{C}^{r\times r}$ and $\boldsymbol{P} \in \mathbb{B}^{r\times r}$ the diagonal matrix with nonzero diagonal entries and the permutation matrix such that $\boldsymbol{C} = \boldsymbol{PD}$. Then:

$$\begin{aligned}(\operatorname{supp}(\boldsymbol{X}), \operatorname{supp}(\boldsymbol{CB})) &= (\operatorname{supp}(\boldsymbol{AC}^{-1}), \operatorname{supp}(\boldsymbol{CB})) \\ &= (\operatorname{supp}(\boldsymbol{AD}^{-1})\boldsymbol{P}^T, \boldsymbol{P}\operatorname{supp}(\boldsymbol{DB})) \text{ since } \boldsymbol{P} \text{ is a permutation matrix} \\ &= (\operatorname{supp}(\boldsymbol{A})\boldsymbol{P}^T, \boldsymbol{P}\operatorname{supp}(\boldsymbol{B})) \text{ since } \boldsymbol{D} \text{ has nonzero diagonal entries.}\end{aligned} \tag{1}$$

But by definition of $\boldsymbol{A}$ and $\boldsymbol{B}$, we have $(\operatorname{supp}(\boldsymbol{A}), \operatorname{supp}(\boldsymbol{B})) \in \hat{\Omega}$, and since $\hat{\Omega}$ is stable by permutation, we have $(\operatorname{supp}(\boldsymbol{A})\boldsymbol{P}^T, \boldsymbol{P}\operatorname{supp}(\boldsymbol{B})) \in \hat{\Omega}$, which shows that $(\operatorname{supp}(\boldsymbol{X}), \operatorname{supp}(\boldsymbol{CB})) \in \hat{\Omega}$. This means that $\operatorname{supp}(\boldsymbol{CB}) \in \Omega_R(\boldsymbol{X})$. Then, because of the equality $\boldsymbol{AB} = \boldsymbol{XY}$ and $\boldsymbol{A} = \boldsymbol{XC}$, we have $\boldsymbol{XCB} = \boldsymbol{XY}$. By condition 2, we obtain $(\boldsymbol{X}, \boldsymbol{CB}) \sim (\boldsymbol{X}, \boldsymbol{Y})$. But $(\boldsymbol{A}, \boldsymbol{B}) \sim (\boldsymbol{AC}^{-1}, \boldsymbol{CB})$ and $(\boldsymbol{AC}^{-1}, \boldsymbol{CB}) = (\boldsymbol{X}, \boldsymbol{CB})$. We finally conclude that $(\boldsymbol{A}, \boldsymbol{B}) \sim (\boldsymbol{X}, \boldsymbol{Y})$. □

Lemma 2.3. *Let $\hat{\boldsymbol{S}} \in \mathbb{B}^{n\times r} \times \mathbb{B}^{r\times m}$ be a fixed pair of supports, and consider a pair of factors $(\boldsymbol{X}, \boldsymbol{Y}) \in \Sigma_{\hat{\boldsymbol{S}}}$. Then, $(\boldsymbol{X}, \boldsymbol{Y})$ is identifiable for the family $[\hat{\boldsymbol{S}}]$ if, and only if, $(\boldsymbol{X}, \boldsymbol{Y})$ is identifiable for the family $\{\hat{\boldsymbol{S}}\}$.*

Proof. Suppose that $(\boldsymbol{X}, \boldsymbol{Y})$ is identifiable for the family $\{\hat{\boldsymbol{S}}\}$. Then, let $(\boldsymbol{A}, \boldsymbol{B}) \in \bigcup_{\hat{\boldsymbol{S}}' \in [\hat{\boldsymbol{S}}]} \Sigma_{\boldsymbol{S}'_{\boldsymbol{L}}} \times \Sigma_{\boldsymbol{S}'_{\boldsymbol{R}}}$ verifying $\boldsymbol{AB} = \boldsymbol{XY}$. Then, $(\operatorname{supp}(\boldsymbol{A}), \operatorname{supp}(\boldsymbol{B}))$ is equivalent to $(\boldsymbol{S_L}, \boldsymbol{S_R})$. By definition of support equivalence, there exists a permutation matrix $\boldsymbol{P}$ such that $(\operatorname{supp}(\boldsymbol{A})\boldsymbol{P}, \boldsymbol{P}^T\operatorname{supp}(\boldsymbol{B})) = (\boldsymbol{S_L}, \boldsymbol{S_R})$, meaning that $(\boldsymbol{AP}, \boldsymbol{P}^T\boldsymbol{B}) \in \Sigma_{\hat{\boldsymbol{S}}}$. But $(\boldsymbol{AP})(\boldsymbol{P}^T\boldsymbol{B}) = \boldsymbol{AB} = \boldsymbol{XY}$, so we conclude by assumption that $(\boldsymbol{AP}, \boldsymbol{P}^T\boldsymbol{B}) \sim (\boldsymbol{X}, \boldsymbol{Y})$, and since $(\boldsymbol{AP}, \boldsymbol{P}^T\boldsymbol{B}) \sim (\boldsymbol{A}, \boldsymbol{B})$, we finally obtain $(\boldsymbol{A}, \boldsymbol{B}) \sim (\boldsymbol{X}, \boldsymbol{Y})$. □

B.2 Proofs for Section 3.1 (Linearization of the inverse problem)

Lemma 3.2. *The following assertions are verified:*

1. *for any* $(\boldsymbol{X}, \boldsymbol{Y}) \in \mathbb{C}^{n\times r} \times \mathbb{C}^{r\times m}$, *we have* $(\boldsymbol{I_m} \otimes \boldsymbol{X}) \operatorname{vec}(\boldsymbol{Y}) = \operatorname{vec}(\boldsymbol{XY})$*;*
2. *for any* $\boldsymbol{X} \in \mathbb{C}^{n\times r}$, $\boldsymbol{y} \in \mathbb{C}^{rm}$, *we have* $\operatorname{vec}^{-1}((\boldsymbol{I_m} \otimes \boldsymbol{X})\boldsymbol{y}) = \boldsymbol{X} \operatorname{vec}^{-1}(\boldsymbol{y})$.

Proof. For the first assertion, let $(\boldsymbol{X}, \boldsymbol{Y}) \in \mathbb{C}^{n\times r} \times \mathbb{C}^{r\times m}$. Then, we have:

$$\begin{aligned}\operatorname{vec}(\boldsymbol{XY}) &= \begin{pmatrix}(\boldsymbol{XY})_{\bullet 1}\\ \vdots \\ (\boldsymbol{XY})_{\bullet m}\end{pmatrix} = \begin{pmatrix}\boldsymbol{XY}_{\bullet 1}\\ \vdots \\ \boldsymbol{XY}_{\bullet m}\end{pmatrix} = \begin{pmatrix}\boldsymbol{X} & & (0)\\ & \ddots & \\ (0) & & \boldsymbol{X}\end{pmatrix}\begin{pmatrix}\boldsymbol{Y}_{\bullet 1}\\ \vdots \\ \boldsymbol{Y}_{\bullet m}\end{pmatrix} \\ &= (\boldsymbol{I_m} \otimes \boldsymbol{X}) \operatorname{vec}(\boldsymbol{Y}).\end{aligned} \tag{2}$$

For the second assertion, let $\boldsymbol{X} \in \mathbb{C}^{n\times r}$, $\boldsymbol{y} \in \mathbb{C}^{rm}$. Then, by applying the first assertion to $\boldsymbol{X}$ and $\operatorname{vec}^{-1}(\boldsymbol{y})$, we have $(\boldsymbol{I_m} \otimes \boldsymbol{X}) \operatorname{vec}(\operatorname{vec}^{-1}(\boldsymbol{y})) = \operatorname{vec}(\boldsymbol{X} \operatorname{vec}^{-1}(\boldsymbol{y}))$, which leads to $\operatorname{vec}^{-1}((\boldsymbol{I_m} \otimes \boldsymbol{X})\boldsymbol{y}) = \boldsymbol{X} \operatorname{vec}^{-1}(\boldsymbol{y})$ by applying vec^{-1} to this equality. □

Lemma 3.3. *Let* $\boldsymbol{s}, \boldsymbol{s'} \in \mathbb{B}^p$ *be two vector supports. Then, the following assertions are verified:*

1. $\overline{\Sigma}_{\boldsymbol{s}} - \overline{\Sigma}_{\boldsymbol{s'}} = \overline{\Sigma}_{\boldsymbol{s}\cup\boldsymbol{s'}}$*;*
2. $\Sigma_{\boldsymbol{s}} - \Sigma_{\boldsymbol{s'}} = \overline{\Sigma}_{\boldsymbol{s}\cup\boldsymbol{s'}} \backslash \left(\bigcup_{i\in\boldsymbol{s}\Delta\boldsymbol{s'}} \operatorname{span}(\boldsymbol{e_i})^\perp\right) := E_{\boldsymbol{s},\boldsymbol{s'}}$.

Proof.

1. Let $\boldsymbol{u} \in \overline{\Sigma}_{\boldsymbol{s}}$ and $\boldsymbol{v} \in \overline{\Sigma}_{\boldsymbol{s'}}$. Then, $\operatorname{supp}(\boldsymbol{u}-\boldsymbol{v}) \subseteq \operatorname{supp}(\boldsymbol{u}) \cup \operatorname{supp}(\boldsymbol{v}) \subseteq \boldsymbol{s} \cup \boldsymbol{s'}$, which means that $\boldsymbol{u}-\boldsymbol{v} \in \overline{\Sigma}_{\boldsymbol{s}\cup\boldsymbol{s'}}$. Conversely, let $\boldsymbol{w} \in \overline{\Sigma}_{\boldsymbol{s}\cup\boldsymbol{s'}}$. Define $\boldsymbol{y} \in \overline{\Sigma}_{\boldsymbol{s}}$ such that $\boldsymbol{y}_i = \boldsymbol{w}_i$ for all $i \in \boldsymbol{s}$, and $\boldsymbol{y'} \in \overline{\Sigma}_{\boldsymbol{s'}}$ such that $\boldsymbol{y}'_i = -\boldsymbol{w}_i$ for all $i \in \boldsymbol{s'}\backslash\boldsymbol{s}$ and $\boldsymbol{y}'_i = 0$ for all $i \in \boldsymbol{s'} \cap \boldsymbol{s}$. Then, $\boldsymbol{w} = \boldsymbol{y} - \boldsymbol{y'}$ since:
   - for $i \in [\![p]\!]\backslash(\boldsymbol{s}\cup\boldsymbol{s'})$, $\boldsymbol{w}_i = 0 = \boldsymbol{y}_i - \boldsymbol{y}'_i$;
   - for $i \in \boldsymbol{s}$, $\boldsymbol{w}_i = \boldsymbol{w_i} - 0 = \boldsymbol{y}_i - \boldsymbol{y}'_i$;
   - for $i \in \boldsymbol{s'}\backslash\boldsymbol{s}$, $\boldsymbol{w}_i = 0 - (-\boldsymbol{w}_i) = \boldsymbol{y}_i - \boldsymbol{y}'_i$.

   This shows that $\boldsymbol{w} \in \overline{\Sigma}_{\boldsymbol{s}} - \overline{\Sigma}_{\boldsymbol{s'}}$.

2. Let $\boldsymbol{u} \in \Sigma_{\boldsymbol{s}}$ and $\boldsymbol{v} \in \Sigma_{\boldsymbol{s'}}$. Then, for $i \in [\![p]\!]\backslash(\boldsymbol{s}\cup\boldsymbol{s'})$, we have $\boldsymbol{u}_i - \boldsymbol{v}_i = 0-0 = 0$. For $i \in \boldsymbol{s}\backslash\boldsymbol{s'}$, we have $\boldsymbol{u}_i - \boldsymbol{v}_i = \boldsymbol{u}_i \neq 0$ since $\boldsymbol{u} \in \Sigma_{\boldsymbol{s}}$. Symmetrically, for $i \in \boldsymbol{s'}\backslash\boldsymbol{s}$, we have $\boldsymbol{u}_i - \boldsymbol{v}_i \neq 0$. This means that $\boldsymbol{u} - \boldsymbol{v} \in E_{\boldsymbol{s},\boldsymbol{s'}}$. Conversely, let $\boldsymbol{w} \in E_{\boldsymbol{s},\boldsymbol{s'}}$. Then, define $\boldsymbol{u} \in \Sigma_{\boldsymbol{s}}$ such that:

   $$\forall i \in \boldsymbol{s}, \quad \boldsymbol{u}_i := \begin{cases}\boldsymbol{w}_i & \text{if } i \in \boldsymbol{s}\backslash\boldsymbol{s'}\\ 1 & \text{if } i \in \boldsymbol{s}\cap\boldsymbol{s'} \text{ and } \boldsymbol{w}_i = 0\\ \frac{\boldsymbol{w}_i}{2} & \text{otherwise, } i.e. \text{ when } i \in \boldsymbol{s}\cap\boldsymbol{s'} \text{ and } \boldsymbol{w}_i \neq 0\end{cases}. \tag{3}$$

   Similarly, define $\boldsymbol{v} \in \Sigma_{\boldsymbol{s'}}$ such that:

   $$\forall i \in \boldsymbol{s'}, \quad \boldsymbol{v}_i := \begin{cases}-\boldsymbol{w}_i & \text{if } i \in \boldsymbol{s'}\backslash\boldsymbol{s}\\ 1 & \text{if } i \in \boldsymbol{s'}\cap\boldsymbol{s} \text{ and } \boldsymbol{w}_i = 0\\ -\frac{\boldsymbol{w}_i}{2} & \text{otherwise, } i.e. \text{ when } i \in \boldsymbol{s'}\cap\boldsymbol{s} \text{ and } \boldsymbol{w}_i \neq 0\end{cases}. \tag{4}$$

   Then, one can verify that $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v}$, which means that $\boldsymbol{w} \in \Sigma_{\boldsymbol{s}} - \Sigma_{\boldsymbol{s'}}$.

□

B.3 Proofs for Section 3.2 (Closed family of allowed right supports)

Proposition 3.1. *Let $\Omega_R \subseteq \mathbb{B}^{r\times m}$ be a family of allowed right supports, and $\boldsymbol{X} \in \mathbb{C}^{n\times r}$ a fixed left factor. Then the following assertions are equivalent:*

1. *$(\Omega_R, \boldsymbol{X})$ is globally and exactly right identifiable with extension (Definition 3.3);*
2. *for all $\boldsymbol{S_R}, \boldsymbol{S_R}' \in \Omega_R$, we have $\mathrm{Ker}(\boldsymbol{I_m} \otimes \boldsymbol{X}) \cap \left(\overline{\Sigma}_{\mathrm{vec}(\boldsymbol{S_R})} - \overline{\Sigma}_{\mathrm{vec}(\boldsymbol{S_R}')}\right) \subseteq \{0\}$.*

Proof. After proving Proposition A.1, we can also prove Proposition 3.1 as a corollary of Proposition A.1, by considering the specific case where $\Omega_R = \overline{\Omega}_R$. Indeed, assertion 2 of Proposition A.1 is equivalent to:

$$\mathrm{Ker}(\boldsymbol{I_m} \otimes \boldsymbol{X}) \cap \bigcup_{\boldsymbol{S_R}, \boldsymbol{S'_R} \in \Omega_R} \left(\Sigma_{\mathrm{vec}(\boldsymbol{S_R})} - \Sigma_{\mathrm{vec}(\boldsymbol{S_R}')}\right) \subseteq \{0\} \tag{5}$$

Denote:

$$\mathring{\Omega}_R := \{\boldsymbol{S_R} \in \Omega_R \mid \forall \boldsymbol{S'_R} \in \Omega_R,\ \boldsymbol{S_R} \neq \boldsymbol{S'_R} \Rightarrow \boldsymbol{S_R} \not\subseteq \boldsymbol{S'_R}\}. \tag{6}$$

Then, supposing $\Omega_R = \overline{\Omega}_R$, we have:

$$\begin{aligned}
\bigcup_{\boldsymbol{S_R}, \boldsymbol{S'_R} \in \Omega_R} \left(\Sigma_{\mathrm{vec}(\boldsymbol{S_R})} - \Sigma_{\mathrm{vec}(\boldsymbol{S_R}')}\right) &= \bigcup_{\boldsymbol{S_R}, \boldsymbol{S'_R} \in \overline{\Omega}_R} \left(\Sigma_{\mathrm{vec}(\boldsymbol{S_R})} - \Sigma_{\mathrm{vec}(\boldsymbol{S_R}')}\right) \text{ since } \Omega_R = \overline{\Omega}_R \\
&= \bigcup_{\tilde{\boldsymbol{S}}_{\boldsymbol{R}}, \tilde{\boldsymbol{S}}'_{\boldsymbol{R}} \in \mathring{\Omega}_R} \bigcup_{\substack{\boldsymbol{S_R} \subseteq \tilde{\boldsymbol{S}}_{\boldsymbol{R}} \\ \boldsymbol{S'_R} \subseteq \tilde{\boldsymbol{S}}'_{\boldsymbol{R}}}} \left(\Sigma_{\mathrm{vec}(\boldsymbol{S_R})} - \Sigma_{\mathrm{vec}(\boldsymbol{S_R}')}\right) \\
&= \bigcup_{\tilde{\boldsymbol{S}}_{\boldsymbol{R}}, \tilde{\boldsymbol{S}}'_{\boldsymbol{R}} \in \mathring{\Omega}_R} \left(\overline{\Sigma}_{\mathrm{vec}(\tilde{\boldsymbol{S}}_{\boldsymbol{R}})} - \overline{\Sigma}_{\mathrm{vec}(\tilde{\boldsymbol{S}}'_{\boldsymbol{R}})}\right) \text{ by definition (2.6)} \\
&= \bigcup_{\tilde{\boldsymbol{S}}_{\boldsymbol{R}}, \tilde{\boldsymbol{S}}'_{\boldsymbol{R}} \in \mathring{\Omega}_R} \bigcup_{\substack{\boldsymbol{S_R} \subseteq \tilde{\boldsymbol{S}}_{\boldsymbol{R}} \\ \boldsymbol{S'_R} \subseteq \tilde{\boldsymbol{S}}'_{\boldsymbol{R}}}} \left(\overline{\Sigma}_{\mathrm{vec}(\boldsymbol{S_R})} - \overline{\Sigma}_{\mathrm{vec}(\boldsymbol{S_R}')}\right) \\
&= \bigcup_{\boldsymbol{S_R}, \boldsymbol{S'_R} \in \overline{\mathring{\Omega}}_R} \left(\overline{\Sigma}_{\mathrm{vec}(\boldsymbol{S_R})} - \overline{\Sigma}_{\mathrm{vec}(\boldsymbol{S_R}')}\right) \\
&= \bigcup_{\boldsymbol{S_R}, \boldsymbol{S'_R} \in \Omega_R} \left(\overline{\Sigma}_{\mathrm{vec}(\boldsymbol{S_R})} - \overline{\Sigma}_{\mathrm{vec}(\boldsymbol{S_R}')}\right) \text{ since } \overline{\mathring{\Omega}}_R = \overline{\Omega}_R = \Omega_R.
\end{aligned} \tag{7}$$

Condition (5) is therefore equivalent to :

$$\mathrm{Ker}(\boldsymbol{I_m} \otimes \boldsymbol{X}) \cap \bigcup_{\boldsymbol{S_R}, \boldsymbol{S'_R} \in \Omega_R} \left(\overline{\Sigma}_{\mathrm{vec}(\boldsymbol{S_R})} - \overline{\Sigma}_{\mathrm{vec}(\boldsymbol{S_R}')}\right) \subseteq \{0\} \tag{8}$$

which is equivalent to condition 2 of Proposition 3.1.

□

Lemma 3.5. *Let $\boldsymbol{S_R} \in \mathbb{B}^{r\times m}$ and $\boldsymbol{X} \in \mathbb{C}^{n\times r}$. The following assertions are equivalent:*

1. *$\mathrm{Ker}(\boldsymbol{I_m} \otimes \boldsymbol{X}) \cap \overline{\Sigma}_{\mathrm{vec}(\boldsymbol{S_R})} = \{0\}$;*
2. *for all $l \in [\![m]\!]$, we have $\mathrm{Ker}(\boldsymbol{X}) \cap \overline{\Sigma}_{(\boldsymbol{S_R})_{\bullet l}} = \{0\}$;*
3. *for all $l \in [\![m]\!]$, we have $\mathrm{Ker}\left(\boldsymbol{X}_{|[\![n]\!] \times (\boldsymbol{S_R})_{\bullet l}}\right) = \{0\}$;*
4. *for all $l \in [\![m]\!]$, the columns $\{\boldsymbol{X}_{\bullet l} \mid l \in (\boldsymbol{S_R})_{\bullet l}\}$ are linearly independent.*

Proof. Suppose 1, and we want to prove 2. Let $l \in [\![m]\!]$, and $\boldsymbol{y} \in \mathrm{Ker}(\boldsymbol{X}) \cap \overline{\Sigma}_{(\boldsymbol{S_R})_{\bullet l}}$. Define $\boldsymbol{Y} \in \overline{\Sigma}_{\boldsymbol{S_R}}$ such that:

$$\forall l' \in [\![m]\!], \quad \boldsymbol{Y}_{\bullet l'} := \begin{cases} \boldsymbol{y} & \text{if } l' = l \\ 0 & \text{otherwise} \end{cases}. \tag{9}$$

Then, by construction, we have $\boldsymbol{XY} = 0$, which means, by Lemma 3.2, that $(\boldsymbol{I_m} \otimes \boldsymbol{X})\,\mathrm{vec}(\boldsymbol{Y}) = 0$. Since $\boldsymbol{Y} \in \overline{\Sigma}_{\boldsymbol{S_R}}$, we have $\mathrm{vec}(\boldsymbol{Y}) \in \overline{\Sigma}_{\mathrm{vec}(\boldsymbol{S_R})}$. This means that $\mathrm{vec}(\boldsymbol{Y}) \in \mathrm{Ker}(\boldsymbol{I_m} \otimes \boldsymbol{X}) \cap \overline{\Sigma}_{\mathrm{vec}(\boldsymbol{S_R})}$, and by assumption 1, $\mathrm{vec}(\boldsymbol{Y}) = 0$, which leads to $\boldsymbol{y} = 0$.

Conversely, suppose 2, and we want to prove 1. Let $\boldsymbol{y} \in \mathrm{Ker}(\boldsymbol{I_m} \otimes \boldsymbol{X}) \cap \overline{\Sigma}_{\mathrm{vec}(\boldsymbol{S_R})}$. Then, we have $(\boldsymbol{I_m} \otimes \boldsymbol{X})\boldsymbol{y} = 0$, which means, by Lemma 3.2, that $\boldsymbol{X}\,\mathrm{vec}^{-1}(\boldsymbol{y}) = 0$. Since $\boldsymbol{y} \in \overline{\Sigma}_{\mathrm{vec}(\boldsymbol{S_R})}$, we have $\mathrm{vec}^{-1}(\boldsymbol{y}) \in \overline{\Sigma}_{\boldsymbol{S_R}}$. Let $l \in [\![m]\!]$. Then, from $\boldsymbol{X}\,\mathrm{vec}^{-1}(\boldsymbol{y}) = 0$, we obtain $\boldsymbol{X}\,\mathrm{vec}^{-1}(\boldsymbol{y})_{\bullet l} = 0$, and from $\mathrm{vec}^{-1}(\boldsymbol{y}) \in \overline{\Sigma}_{\boldsymbol{S_R}}$, we obtain $\mathrm{vec}^{-1}(\boldsymbol{y})_{\bullet l} \in \overline{\Sigma}_{(\boldsymbol{S_R})_{\bullet l}}$. This leads to $\mathrm{vec}^{-1}(\boldsymbol{y})_{\bullet l} \in \mathrm{Ker}(\boldsymbol{X}) \cap \overline{\Sigma}_{(\boldsymbol{S_R})_{\bullet l}}$, and by assumption 2, we conclude that $\mathrm{vec}^{-1}(\boldsymbol{y})_{\bullet l} = 0$. This is true for all $l \in [\![m]\!]$, so $\boldsymbol{y} = 0$.

Condition 3 and condition 4 are reformulations of condition 2. □

Corollary 3.1. *Let $k \in [\![r]\!]$, and denote $\Omega_R := \{\boldsymbol{S} \in \mathbb{B}^{r\times m} \mid \|\boldsymbol{S}_{\bullet j}\|_0 \le k,\ \forall j \in [\![m]\!]\}$ the family of right supports which are k-sparse by column. Then, for any $\boldsymbol{X} \in \mathbb{C}^{n\times r}$, $(\Omega_R, \boldsymbol{X})$ is globally and exactly right identifiable (Definition 3.2) if, and only if, every subset of $\min(2k, r)$ columns of $\boldsymbol{X}$ is linearly independent.*

Proof. We express the set $\mathcal{T} := \{(\boldsymbol{S_R})_{\bullet j} \cup (\boldsymbol{S_R}')_{\bullet j} \mid \boldsymbol{S_R}, \boldsymbol{S}'_{\boldsymbol{R}} \in \Omega_R,\ j \in [\![m]\!]\}$ for the specific family of right supports k-sparse by columns, which is closed. In fact, we have:

$$\mathcal{T} = \{T \in \mathcal{P}([\![r]\!]) \mid \mathrm{card}(T) \le 2k\} \tag{10}$$

Indeed, let $T \in \mathcal{T}$. Then, there exists $\boldsymbol{S_R}, \boldsymbol{S}'_{\boldsymbol{R}} \in \Omega_R,\ j \in [\![m]\!]$ such that $T = (\boldsymbol{S_R})_{\bullet j} \cup (\boldsymbol{S_R}')_{\bullet j}$. Then, $\mathrm{card}(T) \le \mathrm{card}((\boldsymbol{S_R})_{\bullet j}) + \mathrm{card}((\boldsymbol{S_R}')_{\bullet j}) \le 2k$. Conversely, let $T \subseteq [\![r]\!]$ be a subset of indices such that $\mathrm{card}(T) \le 2k$. We then set $\boldsymbol{S_R}, \boldsymbol{S}'_{\boldsymbol{R}} \in \Omega_R$ where $(\boldsymbol{S_R})_{\bullet 1}$ contains the k first indices of T, and $(\boldsymbol{S}'_{\boldsymbol{R}})_{\bullet 1}$ contains the remaining indices, so that $T = (\boldsymbol{S_R})_{\bullet 1} \cup (\boldsymbol{S}'_{\boldsymbol{R}})_{\bullet 1}$. The remaining columns of $\boldsymbol{S_R}$ and $\boldsymbol{S}'_{\boldsymbol{R}}$ are set to zero for instance. This shows $T \in \mathcal{T}$.

Then, we conclude by remarking that the columns $\{\boldsymbol{X}_{\bullet j} \mid j \in j\}$ are linearly independent for all $T \in \mathcal{T}$ if, and only if, every subset of $\min(2k, r)$ columns of $\boldsymbol{X}$ is linearly independent, because for any subset $T \subseteq [\![r]\!]$, when the columns $\{\boldsymbol{X}_{\bullet j} \mid j \in T\}$ are linearly independent, the columns $\{\boldsymbol{X}_{\bullet j} \mid j \in T'\}$ are linearly independent for any subset $T' \subseteq T$. □

Corollary 3.2. *Let $l \in [\![m]\!]$, and denote $\Omega_R := \{\boldsymbol{S} \in \mathbb{B}^{r\times m} \mid \|\boldsymbol{S}_{i\bullet}\|_0 \le l,\ \forall i \in [\![p]\!]\}$ the family of right supports which are k-sparse by row. Then, for any $\boldsymbol{X} \in \mathbb{C}^{n\times r}$, $(\Omega_R, \boldsymbol{X})$ is globally and exactly right identifiable (Definition 3.2) if, and only if, all the columns of $\boldsymbol{X}$ are linearly independent.*

Proof. When considering Ω_R as the family of right supports l-sparse by column, which is closed, we have $[\![r]\!] \in \mathcal{T}$. Indeed, consider $\boldsymbol{S_R} \in \Omega_R$ where the first column $(\boldsymbol{S_R})_{\bullet 1}$ is full of one, and the other columns are full of zero. Then, we have $[\![r]\!] = (\boldsymbol{S_R})_{\bullet 1} \cup (\boldsymbol{S_R})_{\bullet 1} \in \mathcal{T}$. Then, the columns $\{\boldsymbol{X}_{\bullet j} \mid j \in j\}$ are linearly independent for all $T \in \mathcal{T}$ if, and only if, all the columns of $\boldsymbol{X}$ are linearly independent. □

Corollary 3.3. *Let $(k, l) \in [\![r]\!] \times [\![m]\!]$, and denote $\Omega_R := \{\boldsymbol{S} \in \mathbb{B}^{r\times m} \mid \|\boldsymbol{S}_{\bullet j}\|_0 \le k,\ \|\boldsymbol{S}_{i\bullet}\|_0 \le l,\ \forall (i,j) \in [\![r]\!] \times [\![m]\!]\}$ the family of right supports which are k-sparse by column and l-sparse by row. Then, for any $\boldsymbol{X} \in \mathbb{C}^{n\times r}$, $(\Omega_R, \boldsymbol{X})$ is globally and exactly right identifiable (Definition 3.2) if, and only if, every subset of $\min(2k, r)$ columns of $\boldsymbol{X}$ is linearly independent.*

Proof. The proof is the same as the one of Corollary 3.1 given here. □

Corollary 3.4. *Let $s \in [\![rm]\!]$, and denote $\Omega_R := \{\boldsymbol{S} \in \mathbb{B}^{r\times m} \mid \|\boldsymbol{S}\|_0 \le s\}$ the family of right supports which are globally s-sparse. Then, for any $\boldsymbol{X} \in \mathbb{C}^{n\times r}$, $(\Omega_R, \boldsymbol{X})$ is globally and exactly right identifiable (Definition 3.2) if, and only if, every subset of $\min(2s, r)$ columns of $\boldsymbol{X}$ is linearly independent.*

Proof. We express the set $\mathcal{T} := \{(\boldsymbol{S_R})_{\bullet j} \cup (\boldsymbol{S_R}')_{\bullet j} \mid \boldsymbol{S_R}, \boldsymbol{S'_R} \in \Omega_R,\ j \in [\![m]\!]\}$ for the specific family of right supports globally s-sparse, which is closed. In fact, we have:

$$\mathcal{T} = \{T \in \mathcal{P}([\![r]\!]) \mid \operatorname{card}(T) \leq 2s\} \tag{11}$$

Indeed, let $T \in \mathcal{T}$. Then, there exists $\boldsymbol{S_R}, \boldsymbol{S'_R} \in \Omega_R,\ j \in [\![m]\!]$ such that $T = (\boldsymbol{S_R})_{\bullet j} \cup (\boldsymbol{S_R}')_{\bullet j}$. Then, $\operatorname{card}(T) \leq \operatorname{card}((\boldsymbol{S_R})_{\bullet j}) + \operatorname{card}((\boldsymbol{S_R}')_{\bullet j}) \leq \|\boldsymbol{S_R}\|_0 + \|\boldsymbol{S'_R}\|_0 \leq 2s$. Conversely, let $T \subseteq [\![r]\!]$ be a subset of indices such that $\operatorname{card}(T) \leq 2s$. We then set $\boldsymbol{S_R}, \boldsymbol{S'_R} \in \Omega_R$ in the following way:

- if $s \leq r$, then $(\boldsymbol{S_R})_{\bullet 1}$ contains the s first indices of T, and $(\boldsymbol{S'_R})_{\bullet 1}$ contains the remaining indices, so that $T = (\boldsymbol{S_R})_{\bullet 1} \cup (\boldsymbol{S'_R})_{\bullet 1}$;
- otherwise, $(\boldsymbol{S_R})_{\bullet 1}$ contains all the indices in T, which is possible since $\operatorname{card}(T) \leq r < s$, and $(\boldsymbol{S'_R})_{\bullet 1}$ is set to zero, so that $T = (\boldsymbol{S_R})_{\bullet 1} \cup (\boldsymbol{S'_R})_{\bullet 1}$.

The remaining columns of $\boldsymbol{S_R}$ and $\boldsymbol{S'_R}$ are set to zero for instance. This shows $T \in \mathcal{T}$. Then, we conclude similarly to the remark at the end of the proof of Corollary 3.1 given here. □

B.4 Proofs for Section 4.1 (Redundant structure in a pair of supports)

Lemma 4.1. *Let $\hat{\boldsymbol{S}} \in \mathbb{B}^{n\times r} \times \mathbb{B}^{r\times m}$ be a pair of supports. Then, $\hat{\boldsymbol{S}}$ has a redundant structure if, and only if, there exists a permutation matrix $\boldsymbol{P} \in \mathbb{B}^{r\times r}\backslash\{\boldsymbol{I_r}\}$ different from the identity matrix such that $(\boldsymbol{S_L P}, \boldsymbol{P}^T \boldsymbol{S_R}) = (\boldsymbol{S_L}, \boldsymbol{S_R})$.*

Proof. For necessity, suppose that $\hat{\boldsymbol{S}}$ has a redundant structure. We fix $j_1, j_2 \in [\![r]\!]$ such that $j_1 \neq j_2$ and $((\boldsymbol{S_L})_{\bullet j_1}, (\boldsymbol{S_R})_{j_1 \bullet}) = ((\boldsymbol{S_L})_{\bullet j_2}, (\boldsymbol{S_R})_{j_2 \bullet})$. We define the permutation $\sigma : [\![r]\!] \to [\![r]\!]$ where $\sigma(j_1) = j_2$, $\sigma(j_2) = j_1$, and $\sigma(i) = i$ for all $i \in [\![r]\!]\backslash\{j_1, j_2\}$. Then, for all $i \in [\![r]\!]$, we have $\big((\boldsymbol{S_L})_{\bullet \sigma(i)}, (\boldsymbol{S_R})_{\sigma(i)\bullet}\big) = ((\boldsymbol{S_L})_{\bullet i}, (\boldsymbol{S_R})_{i\bullet})$ by construction. Define now the permutation matrix $\boldsymbol{P} \in \mathbb{B}^{r\times r}$ such that for all $i \in [\![r]\!]$, $\boldsymbol{P}_{\bullet i} = \boldsymbol{e_{\sigma(i)}}$. Then, for all $i \in [\![r]\!]$, we have:

$$\begin{aligned}
\big((\boldsymbol{S_L})_{\bullet \sigma(i)}, (\boldsymbol{S_R})_{\sigma(i)\bullet}\big) &= \big(\boldsymbol{S_L e_{\sigma(i)}}, (\boldsymbol{e_{\sigma(i)}})^T \boldsymbol{S_R}\big) \\
&= \big(\boldsymbol{S_L P}_{\bullet i}, (\boldsymbol{P}_{\bullet i})^T \boldsymbol{S_R}\big) \\
&= \big(\boldsymbol{S_L P}_{\bullet i}, (\boldsymbol{P}^T)_{i\bullet} \boldsymbol{S_R}\big) \\
&= \big((\boldsymbol{S_L P})_{\bullet i}, (\boldsymbol{P}^T \boldsymbol{S_R})_{i\bullet}\big).
\end{aligned} \tag{12}$$

This means that for all $i \in [\![r]\!]$, we have $((\boldsymbol{S_L})_{\bullet i}, (\boldsymbol{S_R})_{i\bullet}) = \big((\boldsymbol{S_L})_{\bullet \sigma(i)}, (\boldsymbol{S_R})_{\sigma(i)\bullet}\big) = \big((\boldsymbol{S_L P})_{\bullet i}, (\boldsymbol{P}^T \boldsymbol{S_R})_{i\bullet}\big)$. In other words, we obtain $(\boldsymbol{S_L P}, \boldsymbol{P}^T \boldsymbol{S_R}) = (\boldsymbol{S_L}, \boldsymbol{S_R})$. We conclude with the remark that by construction, $\boldsymbol{P}$ is not the identity matrix.

For sufficiency, we show the contraposition. Suppose that for all $j_1, j_2 \in [\![r]\!]$ such that $j_1 \neq j_2$, we have $((\boldsymbol{S_L})_{\bullet j_1}, (\boldsymbol{S_R})_{j_1\bullet}) \neq ((\boldsymbol{S_L})_{\bullet j_2}, (\boldsymbol{S_R})_{j_2\bullet})$. Let $\boldsymbol{P} \in \mathbb{B}^{r\times r}\backslash\{\boldsymbol{I_r}\}$ be a permutation matrix different from the identity matrix. Define the permutation $\sigma : [\![r]\!] \to [\![r]\!]$ such that for all $i \in [\![r]\!]$, we have $\boldsymbol{P}_{\bullet i} = \boldsymbol{e_{\sigma(i)}}$. Then, we cannot have $\big((\boldsymbol{S_L})_{\bullet \sigma(i)}, (\boldsymbol{S_R})_{\sigma(i)\bullet}\big) = ((\boldsymbol{S_L})_{\bullet i}, (\boldsymbol{S_R})_{i\bullet})$ for all $i \in [\![r]\!]$, since σ is different from the identity function and all the $((\boldsymbol{S_L})_{\bullet i}, (\boldsymbol{S_R})_{i\bullet})_{i=1}^r$ are different. Therefore, there exists $i \in [\![r]\!]$ such that $\big((\boldsymbol{S_L})_{\bullet \sigma(i)}, (\boldsymbol{S_R})_{\sigma(i)\bullet}\big) \neq ((\boldsymbol{S_L})_{\bullet i}, (\boldsymbol{S_R})_{i\bullet})$, which is equivalent to $\big((\boldsymbol{S_L P})_{\bullet i}, (\boldsymbol{P}^T \boldsymbol{S_R})_{i\bullet}\big) \neq ((\boldsymbol{S_L})_{\bullet i}, (\boldsymbol{S_R})_{i\bullet})$ by the computation in (12). In conclusion, $(\boldsymbol{S_L P}, \boldsymbol{P}^T \boldsymbol{S_R}) \neq (\boldsymbol{S_L}, \boldsymbol{S_R})$.

□

B.5 Proofs for Section 4.2 (Rank 1 contributions representation)

Lemma 4.3. *Let $(\boldsymbol{X}, \boldsymbol{Y}), (\boldsymbol{X'}, \boldsymbol{Y'}) \in \mathbb{C}^{n\times r} \times \mathbb{C}^{r\times m}$. Then, $(\boldsymbol{X}, \boldsymbol{Y}) \sim_s (\boldsymbol{X'}, \boldsymbol{Y'})$ if, and only if, $(\boldsymbol{X}_{\bullet i} \boldsymbol{Y}_{i\bullet})_{i=1}^r = (\boldsymbol{X'}_{\bullet i} \boldsymbol{Y'}_{i\bullet})_{i=1}^r$.*

Proof. Suppose that $(\boldsymbol{X},\boldsymbol{Y}) \sim_s (\boldsymbol{X'},\boldsymbol{Y'})$. Then, by definition, there exists a diagonal matrix $\boldsymbol{D} \in \mathbb{C}^{r\times r}$ with nonzero diagonal entries such that $(\boldsymbol{X'},\boldsymbol{Y'}) = (\boldsymbol{X}\boldsymbol{D}, \boldsymbol{D}^{-1}\boldsymbol{Y})$. This means that for all $i \in [\![r]\!]$, $\boldsymbol{X'_{\bullet i}} = \boldsymbol{D}_{ii}\boldsymbol{X}_{\bullet i}$ and $\boldsymbol{Y'_{i\bullet}} = \frac{1}{\boldsymbol{D}_{ii}}\boldsymbol{Y}_{i\bullet}$. Then, we have $\boldsymbol{X'_{\bullet i}}\boldsymbol{Y'_{i\bullet}} = (\boldsymbol{D}_{ii}\boldsymbol{X}_{\bullet i})(\frac{1}{\boldsymbol{D}_{ii}}\boldsymbol{Y}_{i\bullet}) = \boldsymbol{X}_{\bullet i}\boldsymbol{Y}_{i\bullet}$.

Conversely, suppose that $\boldsymbol{X}_{\bullet i}\boldsymbol{Y}_{i\bullet} = \boldsymbol{X'_{\bullet i}}\boldsymbol{Y'_{i\bullet}}$ for all $i \in [\![r]\!]$. Then, by [15, Chapter 7, Lemma 1], there exists, for each $i \in [\![r]\!]$, $D_i \in \mathbb{C}\backslash\{0\}$ such that $\boldsymbol{X'_{\bullet i}} = D_i\boldsymbol{X}_{\bullet i}$ and $\boldsymbol{Y'_{i\bullet}} = \frac{1}{D_i}\boldsymbol{Y'_{i\bullet}}$. By defining $\boldsymbol{D} \in \mathbb{C}^{r\times r}$ the diagonal matrix where $\boldsymbol{D}_{ii} = D_i$ for all $i \in [\![i]\!]$, we show that $(\boldsymbol{X'},\boldsymbol{Y'}) = (\boldsymbol{X}\boldsymbol{D}, \boldsymbol{D}^{-1}\boldsymbol{Y})$, which means that $(\boldsymbol{X},\boldsymbol{Y}) \sim_s (\boldsymbol{X'},\boldsymbol{Y'})$. $\square$

Lemma 4.4. *Let $(\boldsymbol{X},\boldsymbol{Y}), (\boldsymbol{X'},\boldsymbol{Y'}) \in \mathbb{C}^{n\times r} \times \mathbb{C}^{r\times m}$. Then, $(\boldsymbol{X},\boldsymbol{Y}) \sim (\boldsymbol{X'},\boldsymbol{Y'})$ if, and only if, there exists a permutation $\sigma : [\![r]\!] \to [\![r]\!]$ such that $\boldsymbol{X'_{\bullet i}}\boldsymbol{Y'_{i\bullet}} = \boldsymbol{X}_{\bullet\sigma(i)}\boldsymbol{Y}_{\sigma(i)\bullet}$ for all $i \in [\![r]\!]$.*

Proof. Suppose that $(\boldsymbol{X},\boldsymbol{Y}) \sim (\boldsymbol{X'},\boldsymbol{Y'})$. Then, there exists a permutation matrix $\boldsymbol{P} \in \mathbb{B}^{r\times r}$ and a diagonal matrix $\boldsymbol{D} \in \mathbb{C}^{r\times r}$ with nonzero diagonal entries such that $(\boldsymbol{X'},\boldsymbol{Y'}) = (\boldsymbol{X}\boldsymbol{D}\boldsymbol{P}, \boldsymbol{P}^T\boldsymbol{D}^{-1}\boldsymbol{Y})$. We define the permutation $\sigma : [\![r]\!] \to [\![r]\!]$ such that for all $i \in [\![r]\!]$, we have $\boldsymbol{P}_{\bullet i} = \boldsymbol{e_{\sigma(i)}}$. Then, for all $i \in [\![r]\!]$, we have $(\boldsymbol{X'})_{\bullet i} = (\boldsymbol{X}\boldsymbol{D})_{\bullet\sigma(i)}$ and $(\boldsymbol{Y'})_{i\bullet} = (\boldsymbol{D}^{-1}\boldsymbol{Y})_{\sigma(i)\bullet}$. In particular, we have $\boldsymbol{X'_{\bullet i}}\boldsymbol{Y'_{i\bullet}} = (\boldsymbol{X}\boldsymbol{D})_{\bullet\sigma(i)}(\boldsymbol{D}^{-1}\boldsymbol{Y})_{\sigma(i)\bullet} = \boldsymbol{X}_{\bullet\sigma(i)}\boldsymbol{Y}_{\sigma(i)\bullet}$ for all $i \in [\![r]\!]$.

Conversely, suppose that there exists a permutation $\sigma : [\![r]\!] \to [\![r]\!]$ such that $\boldsymbol{X'_{\bullet i}}\boldsymbol{Y'_{i\bullet}} = \boldsymbol{X}_{\bullet\sigma(i)}\boldsymbol{Y}_{\sigma(i)\bullet}$ for all $i \in [\![r]\!]$. Then, define $\boldsymbol{P} \in \mathbb{B}^{r\times r}$ the permutation matrix for which $\boldsymbol{P}_{\bullet i} = \boldsymbol{e_{\sigma(i)}}$ for all $i \in [\![r]\!]$. This means that we have $\boldsymbol{X'_{\bullet i}}\boldsymbol{Y'_{i\bullet}} = \boldsymbol{X}_{\bullet\sigma(i)}\boldsymbol{Y}_{\sigma(i)\bullet} = (\boldsymbol{X}\boldsymbol{P})_{\bullet i}(\boldsymbol{P}^T\boldsymbol{Y})_{i\bullet}$ for all $i \in [\![r]\!]$. By [15, Chapter 7, Lemma 1], for each $i \in [\![r]\!]$, there exists $D_i \in \mathbb{C}\backslash\{0\}$ such that $(\boldsymbol{X'})_{\bullet i} = D_i(\boldsymbol{X}\boldsymbol{P})_{\bullet i}$ and $(\boldsymbol{Y'_{i\bullet}}) = \frac{1}{D_i}(\boldsymbol{P}^T\boldsymbol{Y})_{i\bullet}$. Then, we define the diagonal matrix $\boldsymbol{D} \in \mathbb{C}^{r\times r}$ such that $\boldsymbol{D}_{ii} = D_i$ for all $i \in [\![r]\!]$. This means that $(\boldsymbol{X'},\boldsymbol{Y'}) = (\boldsymbol{X}\boldsymbol{P}\boldsymbol{D}, \boldsymbol{D}^{-1}\boldsymbol{P}\boldsymbol{Y})$, and $(\boldsymbol{X},\boldsymbol{Y}) \sim (\boldsymbol{X'},\boldsymbol{Y'})$. $\square$

Lemma 4.5. *Let $\hat{\boldsymbol{S}} \in \mathbb{B}^{n\times r} \times \mathbb{B}^{r\times m}$. Consider $(\boldsymbol{A},\boldsymbol{B}) \in \mathbb{C}^{n\times r} \times \mathbb{C}^{r\times m}$ a pair of factors. Then, $(\boldsymbol{A}_{\bullet i}\boldsymbol{B}_{i\bullet})_{i=1}^r \in \prod_{i=1}^r \Sigma_{\boldsymbol{S_i}}$ if, and only if, $(\boldsymbol{A},\boldsymbol{B}) \in \Sigma_{\hat{\boldsymbol{S}}}$.*

Proof. If $(\boldsymbol{A}_{\bullet i}\boldsymbol{B}_{i\bullet})_{i=1}^r \in \prod_{i=1}^r \Sigma_{\boldsymbol{S_i}}$, then for $i \in [\![r]\!]$, we have $\operatorname{supp}(\boldsymbol{A}_{\bullet i}\boldsymbol{B}_{i\bullet}) = \boldsymbol{S_i} = (\boldsymbol{S_L})_{\bullet i}(\boldsymbol{S_R})_{i\bullet}$. Since $\boldsymbol{A}_{\bullet i}\boldsymbol{B}_{i\bullet}$ is at most of rank 1, we have $\operatorname{supp}(\boldsymbol{A}_{\bullet i}) = (\boldsymbol{S_L})_{\bullet i}$ and $\operatorname{supp}(\boldsymbol{B}_{i\bullet}) = (\boldsymbol{S_R})_{i\bullet}$ for $i \in [\![r]\!]$. This means that $(\boldsymbol{A},\boldsymbol{B}) \in \Sigma_{\hat{\boldsymbol{S}}}$. Conversely, if $(\boldsymbol{A},\boldsymbol{B}) \in \Sigma_{\hat{\boldsymbol{S}}}$, then for all $i \in [\![r]\!]$, $\operatorname{supp}(\boldsymbol{A}_{\bullet i}) = (\boldsymbol{S_L})_{\bullet i}$ and $\operatorname{supp}(\boldsymbol{B}_{i\bullet}) = (\boldsymbol{S_R})_{i\bullet}$, which means that $\operatorname{supp}(\boldsymbol{A}_{\bullet i}\boldsymbol{B}_{i\bullet}) = (\boldsymbol{S_L})_{\bullet i}(\boldsymbol{S_R})_{i\bullet} = \boldsymbol{S_i}$. $\square$

B.6 Proofs for Section 4.3 (Fixing a pair of support with redundancy)

Proposition 4.1. *Let $\hat{\boldsymbol{S}} \in \mathbb{B}^{n\times r} \times \mathbb{B}^{r\times m}$ be a pair of supports. If $\hat{\boldsymbol{S}}$ has a redundant structure, then $\hat{\boldsymbol{S}}$ is not globally identifiable with fixed support.*

Proof. Since $\hat{\boldsymbol{S}}$ has a redundant structure, there exists, by Lemma 4.6, two indices $j_1, j_2 \in [\![r]\!]$ such that $j_1 \neq j_2$ and $\boldsymbol{S_{j_1}} = \boldsymbol{S_{j_2}}$. For all $i \in [\![r]\!]$, denote the unique vectors $(\boldsymbol{s_i^L}, \boldsymbol{s_i^R}) \in \mathbb{B}^n \times \mathbb{B}^m$ such that $\boldsymbol{S_i} = \boldsymbol{s_i^L}\boldsymbol{s_i^R}$. We also denote $\boldsymbol{S} := \boldsymbol{S_{j_1}} = \boldsymbol{S_{j_2}}$, and $(\boldsymbol{s_L}, \boldsymbol{s_R}) \in \mathbb{B}^n \times \mathbb{B}^m$ such that $\boldsymbol{S} = \boldsymbol{s_L}\boldsymbol{s_R}^T$. Then, define $(\boldsymbol{X},\boldsymbol{Y}), (\boldsymbol{X'},\boldsymbol{Y'}) \in \mathbb{C}^{n\times r} \times \mathbb{C}^{r\times m}$ as:

- $(\boldsymbol{X}_{\bullet j_1}, \boldsymbol{Y}_{j_1\bullet}) = (\boldsymbol{s_L}, \boldsymbol{s_R}^T)$, $(\boldsymbol{X}_{\bullet j_2}, \boldsymbol{Y}_{j_2\bullet}) = (-\boldsymbol{s_L}, \boldsymbol{s_R}^T)$;
- $(\boldsymbol{X'_{\bullet j_1}}, \boldsymbol{Y'_{j_1\bullet}}) = (2\boldsymbol{s_L}, \boldsymbol{s_R}^T)$, $(\boldsymbol{X'_{\bullet j_2}}, \boldsymbol{Y'_{j_2\bullet}}) = (-2\boldsymbol{s_L}, \boldsymbol{s_R}^T)$;
- and $(\boldsymbol{X}_{\bullet i}, \boldsymbol{Y}_{i\bullet}) = (\boldsymbol{X'_{\bullet i}}, \boldsymbol{Y'_{i\bullet}}) = (\boldsymbol{s_i^L}, \boldsymbol{s_i^R}^T)$ for all $i \in [\![r]\!]\backslash\{j_1, j_2\}$.

Then, by construction, we have:

- $\boldsymbol{X}_{\bullet j_1}\boldsymbol{Y}_{j_1\bullet} = \boldsymbol{S}$, $\boldsymbol{X}_{\bullet j_2}\boldsymbol{Y}_{j_2\bullet} = -\boldsymbol{S}$;
- $\boldsymbol{X'_{\bullet j_1}}\boldsymbol{Y'_{j_1\bullet}} = 2\boldsymbol{S}$, $\boldsymbol{X'_{\bullet j_2}}\boldsymbol{Y'_{j_2\bullet}} = -2\boldsymbol{S}$;
- and $\boldsymbol{X}_{\bullet i}\boldsymbol{Y}_{i\bullet} = \boldsymbol{X'_{\bullet i}}\boldsymbol{Y'_{i\bullet}} = \boldsymbol{S_i}$ for all $i \in [\![r]\!]\backslash\{j_1, j_2\}$.

This leads to $\sum_{i=1}^r \boldsymbol{X}_{\bullet i}\boldsymbol{Y}_{i\bullet} = \sum_{i=1}^r \boldsymbol{X'}_{\bullet i}\boldsymbol{Y'}_{i\bullet}$, which means that $\boldsymbol{XY} = \boldsymbol{X'Y'}$. And by construction, we have $(\boldsymbol{X},\boldsymbol{Y}),(\boldsymbol{X'},\boldsymbol{Y'}) \in \Sigma_{\hat{\boldsymbol{S}}}$ by Lemma 4.5. Now, let us show that $(\boldsymbol{X},\boldsymbol{Y})$ is not equivalent to $(\boldsymbol{X'},\boldsymbol{Y'})$. Let $\sigma : [\![r]\!] \to [\![r]\!]$ be any permutation. Denote $j_0 := \sigma^{-1}(j_1)$. Then, we have $\boldsymbol{X}_{\bullet j_0}\boldsymbol{Y}_{j_0\bullet} \neq \boldsymbol{X'}_{\bullet\sigma(j_0)}\boldsymbol{Y'}_{\sigma(j_0)\bullet}$. Indeed, $\boldsymbol{X'}_{\bullet\sigma(j_0)}\boldsymbol{Y'}_{\sigma(j_0)\bullet} = \boldsymbol{X'}_{\bullet j_1}\boldsymbol{Y'}_{j_1\bullet} = 2\boldsymbol{S}$, and:

- if $j_0 = j_1$, then we have $\boldsymbol{X}_{\bullet j_0}\boldsymbol{Y}_{j_0\bullet} = \boldsymbol{S}$;
- if $j_0 = j_2$, then we have $\boldsymbol{X}_{\bullet j_0}\boldsymbol{Y}_{j_0\bullet} = -\boldsymbol{S}$;
- if $j_0 \in [\![r]\!]\backslash\{j_1,j_2\}$, then we have $\boldsymbol{X}_{\bullet j_0}\boldsymbol{Y}_{j_0\bullet} = \boldsymbol{S}_{\boldsymbol{j_0}}$.

This shows, by contraposition of Lemma 4.4, that $(\boldsymbol{X},\boldsymbol{Y})$ is not equivalent to $(\boldsymbol{X'},\boldsymbol{Y'})$. □

B.7 Proofs for Section 4.4 (Fixing a pair of supports without symmetry)

Proposition 4.2. *Let $\hat{\boldsymbol{S}} \in \mathbb{B}^{n\times r}\times\mathbb{B}^{r\times m}$ be a pair of supports. Then, $\hat{\boldsymbol{S}}$ is globally identifiable with fixed support up to scaling if, and only if,:*

$$\mathcal{N}(\mathscr{S}) \cap \prod_{i=1}^{r} \Delta_{\boldsymbol{S}_{\boldsymbol{i}},1} = \{0\}. \tag{14}$$

Proof. For sufficiency, suppose that $\mathcal{N}(\mathscr{S}) \cap \prod_{i=1}^r \Delta_{\boldsymbol{S}_{\boldsymbol{i}},1} = \{0\}$. Let $(\boldsymbol{X},\boldsymbol{Y}),(\boldsymbol{X'},\boldsymbol{Y'}) \in \Sigma_{\hat{\boldsymbol{S}}}$ such that $\boldsymbol{XY} = \boldsymbol{X'Y'}$. Then, by Lemma 4.5, for all $i \in [\![r]\!]$, we have $\boldsymbol{X}_{\bullet i}\boldsymbol{Y}_{i\bullet}, \boldsymbol{X'}_{\bullet i}\boldsymbol{Y'}_{i\bullet} \in \Sigma_{\boldsymbol{S}_{\boldsymbol{i}},1}$, which means that $\boldsymbol{X}_{\bullet i}\boldsymbol{Y}_{i\bullet} - \boldsymbol{X'}_{\bullet i}\boldsymbol{Y'}_{i\bullet} \in \Delta_{\boldsymbol{S}_{\boldsymbol{i}},1}$. But we also have $\mathscr{S}((\boldsymbol{X}_{\bullet i}\boldsymbol{Y}_{i\bullet} - \boldsymbol{X'}_{\bullet i}\boldsymbol{Y'}_{i\bullet})_{i=1}^r) = \boldsymbol{XY} - \boldsymbol{X'Y'} = 0$. Therefore, we conclude that $(\boldsymbol{X}_{\bullet i}\boldsymbol{Y}_{i\bullet} - \boldsymbol{X'}_{\bullet i}\boldsymbol{Y'}_{i\bullet})_{i=1}^r \in \mathcal{N}(\mathscr{S}) \cap \prod_{i=1}^r \Delta_{\boldsymbol{S}_{\boldsymbol{i}},1} = \{0\}$, which means $\boldsymbol{X}_{\bullet i}\boldsymbol{Y}_{i\bullet} = \boldsymbol{X'}_{\bullet i}\boldsymbol{Y'}_{i\bullet}$ for all $i \in [\![r]\!]$, and $(\boldsymbol{X},\boldsymbol{Y}) \sim_s (\boldsymbol{X'},\boldsymbol{Y'})$ by Lemma 4.3.

For necessity, suppose that $\hat{\boldsymbol{S}}$ is globally identifiable with fixed support up to scaling. Let $\underline{\boldsymbol{C}} \in \mathcal{N}(\mathscr{S}) \cap \prod_{i=1}^r \Delta_{\boldsymbol{S}_{\boldsymbol{i}},1}$. Since $\prod_{i=1}^r \Delta_{\boldsymbol{S}_{\boldsymbol{i}},1} = \prod_{i=1}^r \Sigma_{\boldsymbol{S}_{\boldsymbol{i}},1} - \prod_{i=1}^r \Sigma_{\boldsymbol{S}_{\boldsymbol{i}},1}$, we can decompose $\underline{\boldsymbol{C}} = \underline{\boldsymbol{C}}^{(\boldsymbol{1})} - \underline{\boldsymbol{C}}^{(\boldsymbol{2})}$ where $\underline{\boldsymbol{C}}^{(\boldsymbol{1})}, \underline{\boldsymbol{C}}^{(\boldsymbol{2})} \in \prod_{i=1}^r \Sigma_{\boldsymbol{S}_{\boldsymbol{i}},1}$. Then we have $0 = \mathscr{S}(\underline{\boldsymbol{C}}) = \mathscr{S}(\underline{\boldsymbol{C}}^{(\boldsymbol{1})}) - \mathscr{S}(\underline{\boldsymbol{C}}^{(\boldsymbol{2})})$. But for $k \in \{1,2\}$, since $\underline{\boldsymbol{C}}^{(\boldsymbol{k})} \in (\mathcal{R}_1)^r$, there exists, for $i \in [\![r]\!]$, $\boldsymbol{a}_{\boldsymbol{i}}^{(\boldsymbol{k})} \in \mathbb{C}^n$ and $\boldsymbol{b}_{\boldsymbol{i}}^{(\boldsymbol{k})} \in \mathbb{C}^m$ such that $\boldsymbol{a}_{\boldsymbol{i}}^{(\boldsymbol{k})}\boldsymbol{b}_{\boldsymbol{i}}^{(\boldsymbol{k})^T} = \boldsymbol{C}_{\boldsymbol{i}}^{(\boldsymbol{k})}$. We then define $(\boldsymbol{A}^{(\boldsymbol{k})},\boldsymbol{B}^{(\boldsymbol{k})}) \in \mathbb{C}^{n\times r}\times\mathbb{C}^{r\times m}$ such that $\boldsymbol{A}_{\bullet i}^{(\boldsymbol{k})} = \boldsymbol{a}_{\boldsymbol{i}}^{(\boldsymbol{k})}$ and $\boldsymbol{B}_{i\bullet}^{(\boldsymbol{k})} = \boldsymbol{b}_{\boldsymbol{i}}^{(\boldsymbol{k})^T}$, for $k \in \{1,2\}$. Therefore, the equality $\boldsymbol{A}^{(\boldsymbol{1})}\boldsymbol{B}^{(\boldsymbol{1})} = \boldsymbol{A}^{(\boldsymbol{2})}\boldsymbol{B}^{(\boldsymbol{2})}$ is verified. Moreover, by Lemma 4.5, we have $(\boldsymbol{A}^{(\boldsymbol{k})},\boldsymbol{B}^{(\boldsymbol{k})}) \in \Sigma_{\hat{\boldsymbol{S}}}$ for $k \in \{1,2\}$. This means, by assumption, that $(\boldsymbol{A}^{(\boldsymbol{1})},\boldsymbol{B}^{(\boldsymbol{1})}) \sim_s (\boldsymbol{A}^{(\boldsymbol{2})},\boldsymbol{B}^{(\boldsymbol{2})})$, and $\boldsymbol{A}_{\bullet i}^{(\boldsymbol{1})}\boldsymbol{B}_{i\bullet}^{(\boldsymbol{1})^T} = \boldsymbol{A}_{\bullet i}^{(\boldsymbol{2})}\boldsymbol{B}_{i\bullet}^{(\boldsymbol{2})^T}$ for all $i \in [\![r]\!]$ by Lemma 4.3. Therefore, $\underline{\boldsymbol{C}}^{(\boldsymbol{1})} = \underline{\boldsymbol{C}}^{(\boldsymbol{2})}$, and $\underline{\boldsymbol{C}} = 0$. □

Proposition 4.4. *Let $\underline{\boldsymbol{S}} \in (\mathbb{B}^{n\times m}\cap\mathcal{R}_1)^2$ be a pair of rank 1 supports. If $\mathcal{N}(\mathscr{S}) \cap \prod_{i=1}^2 \Delta_{\boldsymbol{S}_{\boldsymbol{i}},1} = \{0\}$, then $\underline{\boldsymbol{S}}$ is iteratively completable from observable supports.*

Proof. We will show the contraposition of the lemma. Suppose that $\underline{\boldsymbol{S}}$ is not iteratively completable from observable support, which means, by Lemma 4.12, that $\boldsymbol{S_1}$ is not completable from $\boldsymbol{S_1}\backslash\boldsymbol{S_2}$ and $\boldsymbol{S_2}$ is not completable from $\boldsymbol{S_2}\backslash\boldsymbol{S_1}$. For $i \in [\![2]\!]$, denote $(\boldsymbol{s}_{\boldsymbol{i}}^{\boldsymbol{L}}, \boldsymbol{s}_{\boldsymbol{i}}^{\boldsymbol{R}}) \in \mathbb{B}^n\times\mathbb{B}^m$ such that $\boldsymbol{S_i} = \boldsymbol{s}_{\boldsymbol{i}}^{\boldsymbol{L}}\boldsymbol{s}_{\boldsymbol{i}}^{\boldsymbol{R}^T}$. Then, by contraposition of Lemma 4.9, we have $(\boldsymbol{s}_{\boldsymbol{1}}^{\boldsymbol{L}} \subseteq \boldsymbol{s}_{\boldsymbol{2}}^{\boldsymbol{L}}$ or $\boldsymbol{s}_{\boldsymbol{1}}^{\boldsymbol{R}} \subseteq \boldsymbol{s}_{\boldsymbol{2}}^{\boldsymbol{R}})$ and $(\boldsymbol{s}_{\boldsymbol{2}}^{\boldsymbol{L}} \subseteq \boldsymbol{s}_{\boldsymbol{1}}^{\boldsymbol{L}}$ or $\boldsymbol{s}_{\boldsymbol{2}}^{\boldsymbol{R}} \subseteq \boldsymbol{s}_{\boldsymbol{1}}^{\boldsymbol{R}})$. Then, we define $\boldsymbol{C} \in \mathbb{C}^{n\times m}$ as:

$$\forall (k,l) \in [\![n]\!]\times[\![m]\!], \quad (\boldsymbol{C})_{kl} = \begin{cases} 1 & \text{if } (k,l) \in \boldsymbol{S_i}\cap\boldsymbol{S_j} \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

and we set $\boldsymbol{X_1} := \boldsymbol{C}$ and $\boldsymbol{X_2} := -\boldsymbol{C}$. We show that $\boldsymbol{S_1} + \boldsymbol{C} \in \Sigma_{\boldsymbol{S_1},1}$.

1. Firstly, we have $\operatorname{supp}(\boldsymbol{S_1}+\boldsymbol{C}) = \operatorname{supp}(\boldsymbol{S_1})$.

2. Secondly, if $s_1^L \subseteq s_2^L$, then $s_1^L \cap s_2^L = s_1^L$, and we verify that the nonzero columns of $(S_1 + C)$ are colinear:

   - for $j \in s_1^R \cap s_2^R$, we have $(S_1 + C)_{\bullet j} = (S_1)_{\bullet j} + (C)_{\bullet j} = 2(S_1)_{\bullet j} = 2s_1^L$,
   - and for $j \in s_1^R \backslash s_2^R$, we have $(S_1 + C)_{\bullet j} = (S_1)_{\bullet j} = s_1^L$.

   Otherwise, if $s_1^R \subseteq s_2^R$, then $s_1^R \cap s_2^R = s_1^R$, and we verify similarly that the nonzero rows of $(S_1 + C)$ are colinear.

In conclusion, $(S_1 + C) \in \Sigma_{S_1,1}$. We use the same argument to show that $(S_2 + C) \in \Sigma_{S_2,1}$. Finally, this means that $X_1 = (S_1 + C) - (S_1) \in \Delta_{S_1,1}$, and $X_2 = -(S_2 + C) + (S_2) \in \Delta_{S_2,1}$. In conclusion, we obtain $(X_1, X_2) \in \mathcal{N}(\mathscr{S}) \cap \prod_{i=1}^2 \Delta_{S_i,1}$, but $(X_1, X_2) \neq 0$, which ends the proof. $\square$

B.8 Proofs for Section A.1 (General family of allowed right supports)

Lemma A.3. *Let $S_R, S_R' \in \mathbb{B}^{r \times m}$ be two right supports, and $X \in \mathbb{C}^{n \times r}$ a fixed left factor. Suppose that $S_R \neq S_R'$. Then, the following assertions are equivalent:*

1. $\mathrm{Ker}(I_m \otimes X) \cap \overline{\Sigma}_{\mathrm{vec}(S_R) \cup \mathrm{vec}(S_R')} \subseteq \bigcup_{i \in \mathrm{vec}(S_R) \Delta \mathrm{vec}(S_R')} \mathrm{span}(e_i)^\perp$, *where $(e_i)_{i=1}^{rm}$ is the canonical basis of $\mathbb{C}^{rm}$;*

2. *there exists an index $i \in \mathrm{vec}(S_R) \Delta \mathrm{vec}(S_R')$ such that:*

$$\mathrm{Ker}(I_m \otimes X) \cap \overline{\Sigma}_{\mathrm{vec}(S_R) \cup \mathrm{vec}(S_R')} \subseteq \mathrm{span}(e_i)^\perp. \tag{8}$$

Proof. The implication 2 $\Rightarrow$ 1 is given by definition of the union operator. Now, we show the contraposition of the converse: suppose 2 isn't verified, and let us show that 1 isn't verified. Denote $F := \mathrm{Ker}(I_m \otimes X) \cap \overline{\Sigma}_{\mathrm{vec}(S_R) \cup \mathrm{vec}(S_R')}$ which is a linear subspace of $\mathbb{C}^{rm}$. Then, for each $i \in \mathrm{vec}(S_R) \Delta \mathrm{vec}(S_R')$, we can find a vector $y^{(i)} \in F \setminus (\mathrm{span}(e_i))^\perp$. Denote $I := \mathrm{vec}(S_R) \Delta \mathrm{vec}(S_R')$, which is not empty since $S_R \neq S_R'$.

For all $n \in [\![\mathrm{card}(I)]\!]$, we denote H_n the assertion: "there exists $(\lambda_t)_{t=1}^n \in \mathbb{C}^n$ and a subset of indices $\{i_t\}_{t=1}^n \subseteq I$ such that $\sum_{t=1}^n \lambda_t y^{(i_t)} \in F \setminus \left(\bigcup_{t=1}^n \mathrm{span}(e_{i_t})^\perp\right)$". H_1 is true, because for any $i_1 \in I$, we have by construction $y^{(i_1)} \in F \setminus \left(\mathrm{span}(e_{i_1})^\perp\right)$. Let $n \in [\![\mathrm{card}(I) - 1]\!]$, suppose H_n, and let us show H_{n+1}. By assumption, there exists $(\lambda_t)_{t=1}^n \in \mathbb{C}^n$ and a subset of indices $\{i_t\}_{t=1}^n \subseteq I$ such that $\sum_{t=1}^n \lambda_t y^{(i_t)} \in F \setminus \left(\bigcup_{t=1}^n \mathrm{span}(e_{i_t})^\perp\right)$. We denote $v := \sum_{t=1}^n \lambda_t y^{(i_t)}$. Let $i_{n+1} \in I \backslash \{i_t\}_{t=1}^n$. Since $y^{(i_{n+1})} \in F \setminus \left(\mathrm{span}(e_{i_{n+1}})^\perp\right)$, by Lemma A.2, there exists $\lambda_{n+1} \in \mathbb{C}$ such that $\sum_{t=1}^{n+1} \lambda_t y^{(i_t)} = v + \lambda_{n+1} y^{(i_{n+1})} \in F \setminus \left(\bigcup_{t=1}^{n+1} \mathrm{span}(e_{i_t})^\perp\right)$. This shows H_{n+1} and ends the recursion.

In conclusion, $H_{\mathrm{card}(I)}$ is true, which shows the non-inclusion $\mathrm{Ker}(I_m \otimes X) \cap \overline{\Sigma}_{\mathrm{vec}(S_R) \cup \mathrm{vec}(S_R')} \nsubseteq \bigcup_{i \in \mathrm{vec}(S_R) \Delta \mathrm{vec}(S_R')} \mathrm{span}(e_i)^\perp$. $\square$

Lemma A.5. *Let $S_R, S_R' \in \mathbb{B}^{r \times m}$ be two right supports, and $X \in \mathbb{C}^{n \times r}$ a fixed left factor. Suppose that $S_R \neq S_R'$. Then, the following assertions are equivalent:*

1. *there exists an index $i \in \mathrm{vec}(S_R) \Delta \mathrm{vec}(S_R')$ such that the i-th column of $(I_m \otimes X)$ is linearly independent from the columns $\left\{(I_m \otimes X)_{\bullet i'} \mid i' \in (\mathrm{vec}(S_R) \cup \mathrm{vec}(S_R)) \backslash \{i\}\right\}$;*

2. *there exists a column index $l \in [\![m]\!]$ and a row index $k \in (S_R)_{\bullet l} \Delta (S_R')_{\bullet l}$ such that the k-th column of X is linearly independent from the columns $\left\{X_{\bullet j} \mid j \in ((S_R)_{\bullet l} \cup (S_R')_{\bullet l}) \backslash \{k\}\right\}$.*

Proof. Denote $D := \operatorname{vec}(\boldsymbol{S_R}) \Delta \operatorname{vec}(\boldsymbol{S_R}')$, and $T := \operatorname{vec}(\boldsymbol{S_R}) \cup \operatorname{vec}(\boldsymbol{S_R}')$. For any $i \in [\![rm]\!]$, denote $(k_i, l_i) \in [\![r]\!] \times [\![m]\!]$ the unique couple such that $i = (l_i - 1)r + k_i$, given by euclidean division of i by r. For any $l \in [\![m]\!]$, denote also $J(l) := [\![(l-1)n+1; ln]\!]$. Then, we have the following equivalences, which are justified one by one below the equation:

$$\begin{aligned}
&\exists i \in D,\ (\boldsymbol{I_m} \otimes \boldsymbol{X})_{\bullet i} \notin \operatorname{span}\left\{(\boldsymbol{I_m} \otimes \boldsymbol{X})_{\bullet i'} \mid i' \in T \backslash \{i\}\right\} \\
\iff &\exists i \in D,\ (\boldsymbol{I_m} \otimes \boldsymbol{X})_{\bullet i} \notin \operatorname{span}\left\{(\boldsymbol{I_m} \otimes \boldsymbol{X})_{\bullet i'} \mid i' \in (T \backslash \{i\}) \cap [\![(l_i-1)r+1; l_i r]\!]\right\} \\
\iff &\exists i \in D,\ ((\boldsymbol{I_m} \otimes \boldsymbol{X})_{\bullet i})_{|J(l_i)} \notin \operatorname{span}\left\{((\boldsymbol{I_m} \otimes \boldsymbol{X})_{\bullet i'})_{|J(l_i)} \mid i' \in (T \backslash \{i\}) \cap [\![(l_i-1)r+1; l_i r]\!]\right\} \\
\iff &\exists i \in D,\ ((\boldsymbol{I_m} \otimes \boldsymbol{X})_{\bullet i})_{|J(l_i)} \notin \operatorname{span}\left\{((\boldsymbol{I_m} \otimes \boldsymbol{X})_{\bullet (l_i-1)r+j})_{|J(l_i)} \mid j \in ((\boldsymbol{S_R})_{\bullet l_i} \cup (\boldsymbol{S'_R})_{\bullet l_i}) \backslash \{k_i\}\right\} \\
\iff &\exists i \in D,\ \boldsymbol{X}_{\bullet k_i} \notin \left\{\boldsymbol{X}_{\bullet j} \mid j \in ((\boldsymbol{S_R})_{\bullet l_i} \cup (\boldsymbol{S'_R})_{\bullet l_i}) \backslash \{k_i\}\right\} \\
\iff &\exists (k,l) \in \{(k',l') \in [\![r]\!] \times [\![m]\!] \mid (l'-1)r + k' \in D\},\ \boldsymbol{X}_{\bullet k} \notin \operatorname{span}\left\{\boldsymbol{X}_{\bullet j} \mid j \in ((\boldsymbol{S_R})_{\bullet l} \cup (\boldsymbol{S'_R})_{\bullet l}) \backslash \{k\}\right\} \\
\iff &\exists (k,l) \in \bigcup_{l' \in [\![m]\!]} \{(k',l') \mid k' \in (\boldsymbol{S_R})_{\bullet l'} \Delta (\boldsymbol{S'_R})_{\bullet l'}\},\ \boldsymbol{X}_{\bullet k} \notin \operatorname{span}\left\{\boldsymbol{X}_{\bullet j} \mid j \in ((\boldsymbol{S_R})_{\bullet l} \Delta (\boldsymbol{S'_R})_{\bullet l}) \backslash \{k\}\right\} \\
\iff &\exists l \in [\![m]\!],\ \exists k \in (\boldsymbol{S_R})_{\bullet l} \Delta (\boldsymbol{S_R}')_{\bullet l},\ \boldsymbol{X}_{\bullet k} \notin \operatorname{span}\left\{\boldsymbol{X}_{\bullet j} \mid j \in ((\boldsymbol{S_R})_{\bullet l} \Delta (\boldsymbol{S'_R})_{\bullet l}) \backslash \{k\}\right\}.
\end{aligned} \tag{14}$$

We explain here the previous equivalences:

1. the first equivalence comes from the fact that, because of the block structure of $(\boldsymbol{I_m} \otimes \boldsymbol{X})$, we have $\operatorname{supp}((\boldsymbol{I_m} \otimes \boldsymbol{X})_{\bullet i}) \subseteq J_l$, and for all other column indices $i' \in [\![rm]\!] \setminus [\![(l_i-1)r+1; l_i r]\!]$, we have $((\boldsymbol{I_m} \otimes \boldsymbol{X})_{\bullet i'})_{|J(l_i)} = 0$;

2. the second equivalence comes from the fact that, because of the block structure of $(\boldsymbol{I_m} \otimes \boldsymbol{X})$, for all column indices $i' \in [\![(l_i-1)r+1; l_i r]\!]$, we have $\operatorname{supp}((\boldsymbol{I_m} \otimes \boldsymbol{X})_{\bullet i}) \subseteq J(l_i)$;

3. the third equivalence comes from the fact that:
$$(T \backslash \{i\}) \cap [\![(l_i-1)r+1; l_i r]\!] = \left\{(l_i-1)r + j \mid j \in ((\boldsymbol{S_R})_{\bullet l} \cup (\boldsymbol{S'_R})_{\bullet l}) \backslash \{k_i\}\right\} \tag{15}$$
by definition of the vectorization operator given by (3.1);

4. the fourth equivalence comes from the fact that $((\boldsymbol{I_m} \otimes \boldsymbol{X})_{\bullet i'})_{|J(l_i)} = \boldsymbol{X}_{\bullet k_{i'}}$ for all $i' \in [\![(l_i-1)r+1; l_i r]\!]$;

5. the fifth equivalence comes from euclidean division;

6. the sixth equivalence comes from the fact that, for any $l' \in [\![m]\!]$, we have:
$$\operatorname{vec}(\boldsymbol{S_R}) \Delta \operatorname{vec}(\boldsymbol{S_R}') \cap [\![(l'-1)r+1; l'r]\!] = \{(l'-1)r + k' \mid k' \in (\boldsymbol{S_R})_{\bullet l'} \Delta (\boldsymbol{S'_R})_{\bullet l'}\} \tag{16}$$
by definition of the vectorization operator given by (3.1);

7. the seventh equivalence is a reformulation using the definition of the union operator.

$\square$

Lemma A.6. *Let $\Omega_R \subseteq \mathbb{B}^{r \times m}$ be a closed family of allowed right supports, in the sense that $\Omega_R = \overline{\Omega}_R$, and $\boldsymbol{X} \in \mathbb{C}^{n \times r}$ be a fixed left factor. Then, the following assertions are equivalent:*

1. *for all $\boldsymbol{S_R} \in \Omega_R$, the columns $\left\{(\boldsymbol{I_m} \otimes \boldsymbol{X})_{\bullet i'} \mid i' \in \operatorname{vec}(\boldsymbol{S_R})\right\}$ are linearly independent, and for all $\boldsymbol{S_R}, \boldsymbol{S'_R} \in \Omega_R$ such that $\boldsymbol{S_R} \neq \boldsymbol{S'_R}$, there exists an index $i \in \operatorname{vec}(\boldsymbol{S_R}) \Delta \operatorname{vec}(\boldsymbol{S_R}')$ such that the i-th column of $(\boldsymbol{I_m} \otimes \boldsymbol{X})$ is linearly independent from the columns $\left\{(\boldsymbol{I_m} \otimes \boldsymbol{X})_{\bullet i'} \mid i' \in \operatorname{vec}(\boldsymbol{S_R}) \cup \operatorname{vec}(\boldsymbol{S_R}) \backslash \{i\}\right\}$;*

2. *for all* $\boldsymbol{S_R}, \boldsymbol{S'_R} \in \Omega_R$, *the columns* $\left\{(\boldsymbol{I_m} \otimes \boldsymbol{X})_{\bullet i'} \mid i' \in \operatorname{vec}(\boldsymbol{S_R}) \cup \operatorname{vec}(\boldsymbol{S'_R})\right\}$ *are linearly independent.*

Proof. We only need to prove the implication 1 ⇒ 2. Let $\boldsymbol{S_R}, \boldsymbol{S'_R} \in \Omega_R$. Denote $T := \operatorname{vec}(\boldsymbol{S_R}) \cup \operatorname{vec}(\boldsymbol{S'_R})$, $D := \operatorname{vec}(\boldsymbol{S_R}) \Delta \operatorname{vec}(\boldsymbol{S'_R})$ and $\boldsymbol{M} := \boldsymbol{I_m} \otimes \boldsymbol{X}$. If $\boldsymbol{S_R} = \boldsymbol{S'_R}$, by assumption 1, the columns $\{\boldsymbol{M}_{\bullet i} \mid i \in T\}$ are linearly independent, which shows 2.

Suppose now that $\boldsymbol{S_R} \neq \boldsymbol{S'_R}$. Then, we have $D \neq \emptyset$, and for $n \in [\![\operatorname{card}(D)]\!]$, denote H_n the assertion: "there exists a subset $\{i_t\}_{t=1}^n \subseteq D$ such that for all $t \in [\![n]\!]$, the column $\boldsymbol{M}_{\bullet i_t}$ is linearly independent from the columns $\left\{\boldsymbol{M}_{\bullet i'} \mid i' \in T \backslash \{i_1, ..., i_t\}\right\}$". H_1 is true, because by assumption 1, there exists $i_1 \in D$ such that the column $\boldsymbol{M}_{\bullet i_1}$ is linearly independent from the columns $\left\{\boldsymbol{M}_{\bullet i'} \mid T \backslash \{i_1\}\right\}$.

Let $n \in [\![\operatorname{card}(D) - 1]\!]$, suppose that H_n is true, and let us show that H_{n+1} is true. We fix $\{i_t\}_{t=1}^n \subseteq D$ such that for all $t \in [\![n]\!]$, the column $\boldsymbol{M}_{\bullet i_t}$ is linearly independent from the columns $\left\{\boldsymbol{M}_{\bullet i'} \mid i' \in T \backslash \{i_1, ..., i_t\}\right\}$. Define $\boldsymbol{S_R^{(n)}}, \boldsymbol{S_R'^{(n)}} \in \mathbb{B}^{r \times m}$ such that:

$$\operatorname{vec}(\boldsymbol{S_R^{(n)}}) = \operatorname{vec}(\boldsymbol{S_R}) \backslash \{i_1, ..., i_n\}, \tag{17}$$

$$\operatorname{vec}(\boldsymbol{S_R'^{(n)}}) = \operatorname{vec}(\boldsymbol{S'_R}) \backslash \{i_1, ..., i_n\}. \tag{18}$$

Then, we have the inclusions $\boldsymbol{S_R^{(n)}} \subseteq \boldsymbol{S_R}$ and $\boldsymbol{S_R'^{(n)}} \subseteq \boldsymbol{S'_R}$, and since Ω_R is closed and $\boldsymbol{S_R}, \boldsymbol{S'_R} \in \Omega_R$, we have $\boldsymbol{S_R^{(n)}}, \boldsymbol{S_R'^{(n)}} \in \Omega_R$. For any sets A, B, C, we have:

$$\begin{aligned}(A \Delta B) \setminus C &= \left[A \backslash B \,\cup\, B \backslash A\right] \setminus C \\ &= \left[(A \backslash B) \setminus C\right] \,\cup\, \left[(B \backslash A) \setminus C\right] \\ &= \left[(A \backslash C) \setminus B\right] \,\cup\, \left[(B \backslash C) \setminus A\right] \\ &= \left[(A \backslash C) \setminus (B \backslash C)\right] \,\cup\, \left[(B \backslash C) \setminus (A \backslash C)\right],\end{aligned} \tag{19}$$

which means in particular that $(\operatorname{vec}(\boldsymbol{S_R}) \Delta \operatorname{vec}(\boldsymbol{S'_R})) \backslash \{i_t\}_{t=1}^n = \operatorname{vec}(\boldsymbol{S_R^{(n)}}) \Delta \operatorname{vec}(\boldsymbol{S_R'^{(n)}})$. Then, $\operatorname{card}\left(\operatorname{vec}(\boldsymbol{S_R^{(n)}}) \Delta \operatorname{vec}(\boldsymbol{S_R'^{(n)}})\right) = \operatorname{card}(D) - n > 0$ since $n \in [\![\operatorname{card}(D) - 1]\!]$. As a consequence, $\boldsymbol{S_R^{(n)}} \neq \boldsymbol{S_R'^{(n)}}$, and by assumption 1, there exists $i_{n+1} \in \operatorname{vec}(\boldsymbol{S_R^{(n)}}) \Delta \operatorname{vec}(\boldsymbol{S_R'^{(n)}}) = D \backslash \{i_t\}_{t=1}^n$ such that the column $\boldsymbol{M}_{\bullet i_{n+1}}$ is linearly independent from the columns $\left\{\boldsymbol{M}_{\bullet i'} \mid i' \in \left(\operatorname{vec}(\boldsymbol{S_R^{(n)}}) \cup \operatorname{vec}(\boldsymbol{S}_{\boldsymbol{R}}^{(n)})\right) \backslash \{i_{n+1}\}\right\}$. Since:

$$\begin{aligned}\left(\operatorname{vec}(\boldsymbol{S_R^{(n)}}) \cup \operatorname{vec}(\boldsymbol{S}_{\boldsymbol{R}}^{(n)})\right) \backslash \{i_{n+1}\} &= \big(\,(\operatorname{vec}(\boldsymbol{S_R}) \backslash \{i_1, ..., i_n\}) \cup (\operatorname{vec}(\boldsymbol{S'_R}) \backslash \{i_1, ..., i_n\})\,\big) \setminus \{i_{n+1}\} \\ &= \big(\operatorname{vec}(\boldsymbol{S_R}) \cup \operatorname{vec}(\boldsymbol{S'_R})\big) \setminus \{i_1, ..., i_n, i_{n+1}\} \\ &= T \backslash \{i_1, ..., i_n, i_{n+1}\},\end{aligned} \tag{20}$$

$\boldsymbol{M}_{\bullet i_{n+1}}$ is linearly independent from the columns $\left\{\boldsymbol{M}_{\bullet i'} \mid i' \in T \backslash \{i_1, ..., i_{n+1}\}\right\}$, which ends the recursion.

Therefore, on the one hand, $H_{\operatorname{card}(D)}$ is true, so for all $i \in D$, the column $\boldsymbol{M}_{\bullet i}$ is linearly independent from $\{\boldsymbol{M}_{\bullet i'} \mid i' \in T \backslash D\}$, and by construction, the columns $\{\boldsymbol{M}_{\bullet i'} \mid i' \in D\}$ are linearly independent. On the other hand, we show that the columns $\{\boldsymbol{M}_{\bullet i'} \mid i' \in T \backslash D\}$ are linearly independent. Define $\tilde{\boldsymbol{S}}_{\boldsymbol{R}} \in \mathbb{B}^{r \times m}$ such that:

$$\operatorname{vec}(\tilde{\boldsymbol{S}}_{\boldsymbol{R}}) = \operatorname{vec}(\boldsymbol{S_R}) \backslash D = \operatorname{vec}(\boldsymbol{S'_R}) \backslash D. \tag{21}$$

Since $\tilde{\boldsymbol{S}}_{\boldsymbol{R}} \subseteq \boldsymbol{S_R}$ and $\boldsymbol{S_R} \in \Omega_R$, we have $\tilde{\boldsymbol{S}}_{\boldsymbol{R}} \in \Omega_R$ because Ω_R is closed. Then, by assumption 1, the columns $\left\{\boldsymbol{M}_{\bullet i'} \mid i' \in \operatorname{vec}(\tilde{\boldsymbol{S}}_{\boldsymbol{R}})\right\}$ are linearly independent. But we have:

$$\begin{aligned}\operatorname{vec}(\tilde{\boldsymbol{S}}_{\boldsymbol{R}}) &= \operatorname{vec}(\boldsymbol{S}_{\boldsymbol{R}})\backslash D \\ &= (\operatorname{vec}(\boldsymbol{S}_{\boldsymbol{R}})\backslash D) \cup (\operatorname{vec}(\boldsymbol{S}'_{\boldsymbol{R}})\backslash D) \\ &= (\operatorname{vec}(\boldsymbol{S}_{\boldsymbol{R}}) \cup \operatorname{vec}(\boldsymbol{S}'_{\boldsymbol{R}}))\backslash D \\ &= T\backslash D,\end{aligned} \tag{22}$$

so we obtain the linear independence of the columns $\{\boldsymbol{M}_{\bullet i'} \mid i' \in T\backslash D\}$. In conclusion, the columns $\{\boldsymbol{M}_{\bullet i'} \mid i' \in T\}$ are linearly independent.

□

B.9 Proofs for Section A.2 (Expression of the restricted rank 2 null space)

Lemma A.11. *Let $\boldsymbol{X} \in \mathbb{C}^{n\times m}$ be a matrix. Suppose that $\boldsymbol{X}$ has no zero row. Then, we have:*

$$\operatorname{Im}(\boldsymbol{X}) \not\subseteq \bigcup_{i=1}^{n} \operatorname{span}(\boldsymbol{e_i})^{\perp} \tag{18}$$

where $(\boldsymbol{e_i})_{i=1}^n$ is the canonical basis of $\mathbb{C}^n$.

Proof. Denote $K = n - \|\boldsymbol{X}_{\bullet 1}\|_0$ the number of zero entries in the first column of $\boldsymbol{X}$. If $K = 0$, then $\boldsymbol{X}_{\bullet 1} \notin \bigcup_{i=1}^n \operatorname{span}(\boldsymbol{e_i})^{\perp}$, which shows the non-inclusion $\operatorname{Im}(\boldsymbol{X}) \not\subseteq \bigcup_{i=1}^n \operatorname{span}(\boldsymbol{e_i})^{\perp}$. We can therefore suppose now that $K \geq 1$. Define, for $r \in [\![K]\!]$, the assertion H_r: "there exists some column indices $(j_k)_{k=1}^r \in [\![m]\!]^r$ and scalars $(\lambda_k)_{k=1}^r \in \mathbb{C}^r$ such that $\|\boldsymbol{X}_{\bullet 1} + \sum_{k=1}^r \lambda_k \boldsymbol{X}_{\bullet j_k}\|_0 \geq \|\boldsymbol{X}_{\bullet 1}\|_0 + r$".

Let us show that H_1 is true. Since $\boldsymbol{X}_{\bullet 1} \in \operatorname{Im}(\boldsymbol{X})$, and $\|\boldsymbol{X}_{\bullet 1}\|_0 = n - K < n$ because $K \geq 1$, we apply Lemma A.10 to obtain the existence of a row index $i \in [\![n]\!] \backslash \operatorname{supp}(\boldsymbol{X}_{\bullet 1})$, a column index $j \in [\![m]\!]$ and a scalar $\lambda \in \mathbb{C}$ such that $\|\boldsymbol{X}_{\bullet 1}\|_0 + 1 \leq \|\boldsymbol{X}_{\bullet 1} + \lambda \boldsymbol{X}_{\bullet j}\|_0$, which shows H_1.

Now, let $r \in [\![K-1]\!]$, and suppose that H_r is true. Let us show that H_{r+1} is true. By assumption, we fix some column indices $(j_k)_{k=1}^r \in [\![m]\!]^r$ and scalars $(\lambda_k)_{k=1}^r \in \mathbb{C}^r$ such that $\|\boldsymbol{X}_{\bullet 1} + \sum_{k=1}^r \lambda_k \boldsymbol{X}_{\bullet j_k}\|_0 \geq \|\boldsymbol{X}_{\bullet 1}\|_0 + r$. Denote here $\boldsymbol{v_r} := \boldsymbol{X}_{\bullet 1} + \sum_{k=1}^r \lambda_k \boldsymbol{X}_{\bullet j_k}$. There are two cases on the value of $\|\boldsymbol{v_r}\|_0$.

- If $\|\boldsymbol{v_r}\|_0 = n$, then we fix $\lambda_{r+1} = 0$ and take any $j_{r+1} \in [\![m]\!]$. Then, we have $\|\boldsymbol{X}_{\bullet 1} + \sum_{k=1}^{r+1} \lambda_k \boldsymbol{X}_{\bullet j_k}\|_0 = \|\boldsymbol{v_r} + \lambda_{r+1}\boldsymbol{X}_{\bullet j_{r+1}}\|_0 = \|\boldsymbol{v_r}\|_0 = n \geq \|\boldsymbol{X}_{\bullet 1}\|_0 + (r+1)$ since $r \leq K-1 = n - \|\boldsymbol{X}_{\bullet 1}\|_0 - 1$. This shows H_{r+1}.

- Otherwise, if $\|\boldsymbol{v_r}\|_0 < n$, then we can apply Lemma A.10, since $\boldsymbol{v_r} \in \operatorname{Im}(\boldsymbol{X})$. Then, there exists a row index $i \in [\![n]\!] \backslash \operatorname{supp}(\boldsymbol{v_r})$, a column index $j_{r+1} \in [\![m]\!]$ and a scalar $\lambda_{r+1} \in \mathbb{C}$ such that $\|\boldsymbol{v_r}\|_0 + 1 \leq \|\boldsymbol{v_r} + \lambda_{r+1}\boldsymbol{X}_{\bullet j_{r+1}}\|_0$. Then, since $\|\boldsymbol{v_r}\|_0 \geq \|\boldsymbol{X}_{\bullet 1}\|_0 + r$ by assumption, we obtain $\|\boldsymbol{X}_{\bullet 1} + \sum_{k=1}^{r+1} \lambda_k \boldsymbol{X}_{\bullet j_k}\|_0 \geq \|\boldsymbol{X}_{\bullet 1}\|_0 + (r+1)$, which shows H_{r+1}.

This ends the recursion, and in particular H_K is true, which means that there exists column indices $(j_k)_{k=1}^K \in [\![m]\!]^K$ and scalars $(\lambda_k)_{k=1}^K \in \mathbb{C}^K$ such that $n \geq \|\boldsymbol{X}_{\bullet 1} + \sum_{k=1}^K \lambda_k \boldsymbol{X}_{\bullet j_k}\|_0 \geq \|\boldsymbol{X}_{\bullet 1}\|_0 + K = n$. Therefore, all the entries of $\boldsymbol{v_K} := \boldsymbol{X}_{\bullet 1} + \sum_{k=1}^K \lambda_k \boldsymbol{X}_{\bullet j_k}$ are nonzero, which means that $\boldsymbol{v_K} \in \operatorname{Im}(\boldsymbol{X}) \backslash \bigcup_{i=1}^n \operatorname{span}(\boldsymbol{e_i})^{\perp}$. This shows the non-inclusion $\operatorname{Im}(\boldsymbol{X}) \not\subseteq \bigcup_{i=1}^n \operatorname{span}(\boldsymbol{e_i})^{\perp}$. □

Proposition A.2. *Let $\boldsymbol{X} \in \mathbb{C}^{n\times m} \cap \mathcal{R}_2$ be a matrix with rank at most 2. Suppose that $\boldsymbol{X}$ has no zero column and no zero row. Then, there exists two rank 1 matrices $\boldsymbol{C}, \boldsymbol{D} \in \mathbb{C}^{n\times m} \cap \mathcal{R}_1$ such that $\operatorname{supp}(\boldsymbol{C}) = \operatorname{supp}(\boldsymbol{D}) = [\![n]\!] \times [\![m]\!]$, and $\boldsymbol{X} = \boldsymbol{C} - \boldsymbol{D}$.*

Proof. Suppose the case where $n = 1$, *i.e.*, $\boldsymbol{X}$ has only one row. Let $\lambda \in \mathbb{C} \backslash (\{0\} \cup \{-\boldsymbol{X}_{1j} \mid j \in [\![m]\!]\})$. Then, define $\boldsymbol{C}, \boldsymbol{D} \in \mathbb{C}^{1\times m}$ as $\boldsymbol{C} := \boldsymbol{X} + \lambda \mathbb{1}^T$ and $\boldsymbol{D} := \lambda \mathbb{1}^T$, where $\mathbb{1} \in \mathbb{C}^m$ is the vector full of ones. By construction, $\boldsymbol{C}$ and $\boldsymbol{D}$ does not have zero entries, and $\boldsymbol{X} = \boldsymbol{C} - \boldsymbol{D}$, which ends the proof for this specific case.

Now suppose the case where $n \geq 2$. In the following, we will denote:

$$\mathcal{F} := \bigcup_{i=1}^{n} \operatorname{span}(\boldsymbol{e_i})^{\perp}, \quad \mathcal{G} := \bigcup_{j=1}^{m} \operatorname{span}(\boldsymbol{X}_{\bullet j}), \tag{23}$$

where $(\boldsymbol{e_i})_{i=1}^{n}$ is the canonical basis of $\mathbb{C}^n$. Since $\boldsymbol{X}$ has no zero row, by Lemma A.11, we have $\operatorname{Im}(\boldsymbol{X}) \not\subseteq \mathcal{F}$. And since $\boldsymbol{X}$ does not have zero columns, in particular, the dimension of the image $\operatorname{Im}(\boldsymbol{X})$ is not zero. We then distinguish two cases depending on the dimension of the image $\operatorname{Im}(\boldsymbol{X})$, for the construction of a pair of independent vectors $(\boldsymbol{a}, \boldsymbol{c})$.

- If $\dim(\operatorname{Im}(\boldsymbol{X})) = 1$, then let $\boldsymbol{u} \in \operatorname{Im}(\boldsymbol{X})\backslash\{0\}$ and $\boldsymbol{v} \in \mathbb{C}^n \backslash \operatorname{Im}(\boldsymbol{X})$, which is possible since $n \geq 2$. We can show that for all $i \in [\![r]\!]$, we have $\dim\big(\operatorname{span}(\boldsymbol{e_i})^{\perp} \cap \operatorname{span}(\boldsymbol{u}, \boldsymbol{v})\big) \leq 1$. Indeed, let $i \in [\![r]\!]$. We have $\dim\big(\operatorname{span}(\boldsymbol{e_i})^{\perp} \cap \operatorname{span}(\boldsymbol{u}, \boldsymbol{v})\big) \leq \dim(\operatorname{span}(\boldsymbol{u}, \boldsymbol{v})) = 2$. But if $\dim\big(\operatorname{span}(\boldsymbol{e_i})^{\perp} \cap \operatorname{span}(\boldsymbol{u}, \boldsymbol{v})\big) = 2$, then we would have $\operatorname{span}(\boldsymbol{e_i})^{\perp} \cap \operatorname{span}(\boldsymbol{u}, \boldsymbol{v}) = \operatorname{span}(\boldsymbol{u}, \boldsymbol{v})$, because $\operatorname{span}(\boldsymbol{e_i})^{\perp} \cap \operatorname{span}(\boldsymbol{u}, \boldsymbol{v}) \subseteq \operatorname{span}(\boldsymbol{u}, \boldsymbol{v})$ and 2 is finite, which would lead to:
$$\operatorname{Im}(\boldsymbol{X}) = \operatorname{span}(\boldsymbol{u}) \subseteq \operatorname{span}(\boldsymbol{u}, \boldsymbol{v}) = \operatorname{span}(\boldsymbol{e_i})^{\perp} \cap \operatorname{span}(\boldsymbol{u}, \boldsymbol{v}) \subseteq \operatorname{span}(\boldsymbol{e_i})^{\perp} \subseteq \mathcal{F}, \tag{24}$$
and a contradiction with the non-inclusion $\operatorname{Im}(\boldsymbol{X}) \not\subseteq \mathcal{F}$. Therefore, $\operatorname{span}(\boldsymbol{u}, \boldsymbol{v}) \backslash (\operatorname{Im}(\boldsymbol{X}) \cup \mathcal{F} \cup \mathcal{G})$ is non empty, because it is equal to $\operatorname{span}(\boldsymbol{u}, \boldsymbol{v}) \backslash \big(\operatorname{Im}(\boldsymbol{X}) \cup \bigcup_{k=1}^{n} \big(\operatorname{span}(\boldsymbol{e_i})^{T} \cap \operatorname{span}(\boldsymbol{u}, \boldsymbol{v})\big) \cup \mathcal{G}\big)$, which is the relative complement of a union of a finite number of dimension 1 subspaces, with respect to a dimension 2 subspace. Then, we define:
$$\begin{cases} \boldsymbol{a} \in \operatorname{span}(\boldsymbol{u}, \boldsymbol{v}) \backslash (\operatorname{Im}(\boldsymbol{X}) \cup \mathcal{F} \cup \mathcal{G}) \\ \boldsymbol{c} \in \operatorname{span}(\boldsymbol{u}, \boldsymbol{v}) \backslash (\operatorname{Im}(\boldsymbol{X}) \cup \mathcal{F} \cup \mathcal{G} \cup \operatorname{span}(\boldsymbol{a})) \end{cases}, \tag{25}$$
where $\boldsymbol{c}$ exists because $\operatorname{span}(\boldsymbol{u}, \boldsymbol{v}) \backslash (\operatorname{Im}(\boldsymbol{X}) \cup \mathcal{F} \cup \mathcal{G} \cup \operatorname{span}(\boldsymbol{a}))$ is not empty for the same reason.

- If $\dim(\operatorname{Im}(\boldsymbol{X})) = 2$, then similarly to the previous paragraph, we can show that for all $i \in [\![n]\!]$, we have $\dim\big(\operatorname{span}(\boldsymbol{e_i})^{\perp} \cap \operatorname{Im}(\boldsymbol{X})\big) \leq 1$. Let $i \in [\![n]\!]$. We have $\dim\big(\operatorname{span}(\boldsymbol{e_i})^{\perp} \cap \operatorname{Im}(\boldsymbol{X})\big) \leq \dim(\operatorname{Im}(\boldsymbol{X})) = 2$. But if $\dim\big(\operatorname{span}(\boldsymbol{e_i})^{\perp} \cap \operatorname{Im}(\boldsymbol{X})\big) = 2$, then we would have $\operatorname{span}(\boldsymbol{e_i})^{\perp} \cap \operatorname{Im}(\boldsymbol{X}) = \operatorname{Im}(\boldsymbol{X})$, because $\operatorname{span}(\boldsymbol{e_i})^{\perp} \cap \operatorname{Im}(\boldsymbol{X}) \subseteq \operatorname{Im}(\boldsymbol{X})$ and 2 is finite, which would lead to:
$$\operatorname{Im}(\boldsymbol{X}) = \operatorname{span}(\boldsymbol{e_i})^{\perp} \cap \operatorname{Im}(\boldsymbol{X}) \subseteq \operatorname{span}(\boldsymbol{e_i})^{\perp} \subseteq \mathcal{F}, \tag{26}$$
and a contradiction with the non-inclusion $\operatorname{Im}(\boldsymbol{X}) \not\subseteq \mathcal{F}$. Therefore, $\operatorname{Im}(\boldsymbol{X}) \backslash (\mathcal{F} \cup \mathcal{G})$ is non empty, because it is equal to $\operatorname{Im}(\boldsymbol{X}) \backslash \big(\bigcup_{k=1}^{n} \big(\operatorname{span}(\boldsymbol{e_i})^{T} \cap \operatorname{Im}(\boldsymbol{X})\big) \cup \mathcal{G}\big)$, which is the relative complement of a union of a finite number of dimension 1 subspaces, with respect to a dimension 2 subspace. Then, we define:
$$\begin{cases} \boldsymbol{a} \in \operatorname{Im}(\boldsymbol{X}) \backslash (\mathcal{F} \cup \mathcal{G}) \\ \boldsymbol{c} \in \operatorname{Im}(\boldsymbol{X}) \backslash (\mathcal{F} \cup \mathcal{G} \cup \operatorname{span}(\boldsymbol{a})) \end{cases}, \tag{27}$$
where $\boldsymbol{c}$ exists because $\operatorname{Im}(\boldsymbol{X}) \backslash (\mathcal{F} \cup \mathcal{G} \cup \operatorname{span}(\boldsymbol{a}))$ is not empty for the same reason.

In both cases, we obtain by construction that $\operatorname{Im}(\boldsymbol{X}) \subseteq \operatorname{span}(\boldsymbol{a}, \boldsymbol{c})$, where $(\boldsymbol{a}, \boldsymbol{c})$ are independent vectors. We can also show that for all $j \in [\![m]\!]$, $\boldsymbol{X}_{\bullet j} \notin \operatorname{span}(\boldsymbol{a}) \cup \operatorname{span}(\boldsymbol{c})$. Indeed, if there exists $j \in [\![m]\!]$ such that $\boldsymbol{X}_{\bullet j} \in \operatorname{span}(\boldsymbol{a}) \cup \operatorname{span}(\boldsymbol{c})$, then it would mean that $\boldsymbol{a} \in \operatorname{span}(\boldsymbol{X}_{\bullet j})$ or $\boldsymbol{c} \in \operatorname{span}(\boldsymbol{X}_{\bullet j})$, because by assumption, $\boldsymbol{X}_{\bullet j} \neq 0$. This would be in contradiction with the fact that by construction, $\boldsymbol{a} \notin \mathcal{G}$ and $\boldsymbol{c} \notin \mathcal{G}$. Therefore, for each column index $j \in [\![m]\!]$, by expressing the vector $\boldsymbol{X}_{\bullet j}$ in the basis $(\boldsymbol{a}, \boldsymbol{c})$, there exists $b_j, d_j \in \mathbb{C}$ such that $\boldsymbol{X}_{\bullet j} = b_j \boldsymbol{a} - d_j \boldsymbol{c}$, with $b_j \neq 0$ and $d_j \neq 0$. By defining $\boldsymbol{b}, \boldsymbol{d} \in \mathbb{C}^m$ as $\boldsymbol{b_j} := b_j$ and $\boldsymbol{d_j} := d_j$ for all $j \in [\![m]\!]$, we can express $\boldsymbol{X}$ as $\boldsymbol{X} = \boldsymbol{a}\boldsymbol{b}^T - \boldsymbol{c}\boldsymbol{d}^T$. By construction, $\boldsymbol{a} \notin \mathcal{F}$ and $\boldsymbol{c} \notin \mathcal{F}$, so all the entries of $\boldsymbol{a}\boldsymbol{b}^T$ and $\boldsymbol{c}\boldsymbol{d}^T$ are nonzero, which ends the proof. $\square$